

Global existence and L^p convergence rates of planar waves for three-dimensional bipolar Euler-Poisson systems

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Abstract: In the paper, we consider a multi-dimensional bipolar hydrodynamic model from semiconductor devices and plasmas. This system takes the form of Euler-Poisson with electric field and frictional damping added to the momentum equations. We show the global existence and L^p convergence rates of planar diffusion waves for multi-dimensional bipolar Euler-Poisson systems when the initial data are near the planar diffusive waves. A frequency decomposition and approximate Green function based on delicate energy method are used to get the optimal decay rates of the planar diffusion waves. To our knowledge, the L^p ($p \in [2, +\infty]$) -convergence rate of planar waves improves the previous results about the L^2 -convergence rates.

Key words: Bipolar Euler-Poisson system, planar wave, approximate Green function, smooth solution, energy estimates.

AMS subject classifications: 35M20, 35Q35, 76W05.

1 Introduction.

In this paper, we consider the following bipolar Euler-poisson system (hydrodynamic model) in three space dimension:

$$\left\{ \begin{array}{l} \partial_t \rho^+ + \operatorname{div}(\rho^+ u^+) = 0, \\ \partial_t (\rho^+ u_i^+) + \operatorname{div}(\rho^+ u_i^+ u^+) + \partial_{x_i} P(\rho^+) = -\rho^+ u_i^+ + \rho^+ \partial_{x_i} \phi, \quad 1 \leq i \leq 3, \\ \partial_t \rho^- + \operatorname{div}(\rho^- u^-) = 0, \\ \partial_t (\rho^- u_i^-) + \operatorname{div}(\rho^- u_i^- u^-) + \partial_{x_i} P(\rho^-) = -\rho^- u_i^- - \rho^- \partial_{x_i} \phi, \quad 1 \leq i \leq 3, \\ \Delta \phi = \rho^+ - \rho^-, \lim_{|x| \rightarrow \infty} |\nabla \phi| = 0, \end{array} \right. \quad (1.1)$$

with initial data

$$(\rho^\pm, u^\pm)(x, 0) = (\rho_0^\pm(x), u_0^\pm(x)), \quad (1.2)$$

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where ρ^\pm are the two particles's densities, $\rho^\pm u^\pm = (\rho^\pm u_1^\pm, \rho^\pm u_2^\pm, \rho^\pm u_3^\pm)$ are current densities, ϕ is the electrostatic potential, and $P(\rho^\pm)$ are pressures. As usual, we assume the pressure $P(\rho)$ be smooth function in a neighborhood of a constant state ρ^* with $P'(\rho) > 0$. The bipolar Euler-Poisson equations are generally used in the description of charged particle fluids, for example, electrons and holes in semiconductor devices, positively and negatively charged ions in a plasma. This model takes an important role in the fields of applied and computational mathematics, and we can see more details in [12, 21, 25] etc..

Due to their physical importance, mathematical complexity and wide range of applications, many efforts were made for the multi-dimensional bipolar hydrodynamic equations from semiconductors or plasmas. Li [18] showed existence and some limit analysis of stationary solutions for the multi-dimensional bipolar Euler-Poisson system. Ali and Jüngel [1], Li and Zhang [16] and Peng and Xu [23] studied the global smooth solutions of the Cauchy problem for multidimensional bipolar hydrodynamic models in the Sobolev space $H^l(\mathbb{R}^d)$ ($l > 1 + \frac{d}{2}$) and in the Besov space, respectively. Ju [13] discussed the global existence of smooth solutions to the initial boundary value problem for the three-dimensional bipolar Euler-Poisson system. Li and Yang [19] and Wu and Wang [27] showed global existence and L^2 decay rate of the smooth solutions to the three dimensional bipolar Euler-Poisson systems when the initial data are small perturbation of the constant stationary solution. Huang, Mei and Wang [7] showed large time behavior of solution to n -dimensional bipolar hydrodynamic model for semiconductors when the initial data are near to the planar diffusion waves. Ali and Chen [2] studied the zero-electron-mass limit in the Euler-Poisson system for both well- and ill-prepared initial data. Lattanzio [15] and Li [17] investigated the relaxation limit of the multi-dimensional bipolar isentropic Euler-Poisson model for semiconductors, respectively. Ju, etc. [14] discussed the quasi-neutral limit of the two-fluid multi-dimensional Euler-Poisson system. Moreover, it is worth to mentioning that there are a lot of reference about the one-dimensional bipolar Euler-Poisson equation, and the interesting reader can refer to [3, 5, 4, 6, 8, 9, 10, 22, 26, 29, 30] and the reference therein. In particular, motivation by [5, 11], Gasser, Hsiao and Li [4] found that the frictional damping is the key to the nonlinear diffusive phenomena of hyperbolic waves, and investigated the diffusion wave phenomena of smooth “small” solutions for the one-dimensional bipolar hydrodynamic model. Huang and Li [6] also studied the large-time behavior and quasi-neutral limit of L^∞ solution of the one-dimensional Euler-Poisson equations for large initial data with vacuum. That is, they showed that the weak entropy solution of the one-dimensional bipolar Euler-Poisson system converges to the nonlinear diffusion waves. Then Huang, Mei and Wang [7] showed the planar diffusive wave stability to n ($n \geq 2$)-dimensional bipolar hydrodynamic model for semiconductors, and obtained the optimal L^2 and L^∞ decay rates. In this paper, we are going to reconsider global existence of the smooth solution for the multi-dimensional bipolar Euler-Poisson systems, in particular, we try to establish the L^p ($p \in [2, +\infty]$) convergence rates of planar waves.

In the following discussion, we assume that the initial data are a small perturbation of the diffusion profile constructed later with small wave strength. Let the initial data $\rho_0^\pm(x)$ be strictly positive and satisfy

$$\lim_{x_1 \rightarrow \pm\infty} \rho_0^\pm(x) = \rho_\pm,$$

where $\rho_\pm > 0$ are two far field constants with $\rho_- \neq \rho_+$. Similar as the consideration of planar

diffusion waves of damped Euler equations in [28, 20], to define the multi-dimensional planar diffusion wave, we first consider the one dimensional diffusion equation

$$\partial_t w = P(w)_{x_1 x_1}, \quad (1.3)$$

which can be derived from the bipolar Euler-Poisson equations with the relaxation terms in one dimensional case by imposing the Darcy's law, cf. [4, 5]. Then a multi-dimensional diffusion wave $w(x, t)$ is a one dimensional profile in multi-dimensional space. That is, $w(x, t) = W(x_1/\sqrt{1+t})$ is a self-similar solution of the equation (1.3) connecting two end states ρ_\pm at $x_1 = \pm\infty$. Denote $\zeta = \frac{x_1}{\sqrt{t+1}}$, then $W(\zeta)$ satisfies

$$-\frac{1}{2}\zeta \partial_\zeta W = \partial_\zeta(P'(W(\zeta))\partial_\zeta W).$$

For simplicity, let the initial velocity $u_0^\pm(x)$ vanish as $x_1 \rightarrow \pm\infty$, that is,

$$\lim_{x_1 \rightarrow \pm\infty} u_0^\pm(x) = 0,$$

which implies that there is no mass flux coming in from $x_1 = \pm\infty$. This assumption could be removed in a technical way similar to the argument for one dimensional problem because the momentum at $x_1 = \pm\infty$ decays exponentially induced by the linear relaxation terms.

We now recall the bipolar Euler-Poisson systems (1.1) in one space dimension:

$$\begin{cases} \partial_t \rho^+ + \partial_{x_1}(\rho^+ u_1^+) = 0, \\ \partial_t(\rho^+ u_1^+) + \partial_{x_1}(\rho^+ u_1^+ u_1^+) + \partial_{x_1}P(\rho^+) = -\rho^+ u_1^+ + \rho^+ E, \\ \partial_t \rho^- + \partial_{x_1}(\rho^- u_1^-) = 0, \\ \partial_t(\rho^- u_1^-) + \partial_{x_1}(\rho^- u_1^- u_1^-) + \partial_{x_1}P(\rho^-) = -\rho^- u_1^- - \rho^- E, \\ \partial_{x_1}E = \rho^+ - \rho^-, \lim_{x_1 \rightarrow -\infty} E(x_1, t) = 0. \end{cases} \quad (1.4)$$

Denote the solution of (1.4) by $(\tilde{\rho}^\pm, \tilde{u}_1^\pm, \tilde{E})(x_1, t)$. When

$$\lim_{x_1 \rightarrow \pm\infty} \tilde{\rho}^\pm(x_1, 0) = \rho_\pm, \quad \lim_{x_1 \rightarrow \pm\infty} \tilde{u}_1^\pm(x_1, 0) = 0,$$

the time-asymptotic behavior of $(\tilde{\rho}^\pm, \tilde{u}_1^\pm, \tilde{\rho}^-, \tilde{u}_1^-)(x_1, t)$ has been studied in [4], which is shown to be a nonlinear diffusion profile governed by Darcy's law. Roughly speaking, the solution $\tilde{\rho}^\pm(x_1, t)$ converge to a same diffusion wave $W(x_1/\sqrt{1+t})$ up to a constant shift in x_1 . Note that more detailed assumptions on the initial data of the one-dimensional problem (1.4) will be specified in Theorem 2.1.

In this paper, we will generalize this time asymptotic behavior towards a planar diffusion wave to three-dimensional case and establish the related L^p ($2 \leq p \leq \infty$) convergence rates.

As in the consideration of planar diffusion waves of damped Euler equation in [28, 20] and of the bipolar Euler-Poisson system in [7], we do not directly compare the solution of the problem (1.1) with the diffusion wave $W(x_1/\sqrt{1+t})$, instead, we will compare it with the solution of one dimensional problem (1.4). For this, without loss of generality, let us first assume the initial density $\tilde{\rho}^\pm(x_1, 0)$ in (1.4) satisfy

$$\int_{-\infty}^{+\infty} (\tilde{\rho}^\pm(x_1, 0) - W(x_1)) dx_1 = 0. \quad (1.5)$$

For the multi-dimensional problem, the shift function $\delta_0(x')$, where we used the notation $x' = (x_2, x_3)$, can be chosen as in [28, 20] such that the initial density function satisfies

$$\int_{-\infty}^{+\infty} (\rho^\pm(x, 0) - W(x_1 + \delta_0^\pm(x'))) dx_1 = 0.$$

Note that $\delta_0^\pm(x')$ is then uniquely determined by

$$\delta_0^\pm(x') = \frac{1}{\rho_+ - \rho_-} \int_{-\infty}^{\infty} (\rho^\pm(x, 0) - W(x_1)) dx_1,$$

for $\rho_- \neq \rho_+$. Moreover, we assume that basically the shift is uniform in directions other than x_1 at infinity, that is,

$$\lim_{|x'| \rightarrow +\infty} \frac{1}{\rho_+ - \rho_-} \int_{-\infty}^{+\infty} (\rho^\pm(x, 0) - W(x_1)) dx_1 = \delta_*^\pm,$$

Note that this assumption simplifies the problem and it remains unsolved for general perturbation when this assumption fails. An immediate consequence of this assumption is that

$$\lim_{|x'| \rightarrow +\infty} \delta_0^\pm(x') = \delta_*^\pm.$$

And for simplicity, we assume $\delta_*^\pm = \delta_*$ be same constants. With these notations, the main purpose here is to show that the solutions (ρ^\pm, u^\pm) of (1.1) converge to $(\bar{\rho}^\pm, \bar{u}^\pm)$ with certain time decay rates, where

$$\begin{cases} \bar{\rho}^\pm(x, t) = \tilde{\rho}^\pm(x_1 + \delta(x', t), t), \\ \bar{u}^\pm(x, t) = (\tilde{u}_1^\pm(x_1 + \delta(x', t), t), 0, 0), \\ \bar{E}(x, t) = (\tilde{E}(x_1 + \delta(x', t), t), 0, 0), \\ \delta(x', t) = \delta_* + e^{-t}(\delta_0(x') - \delta_*), \end{cases} \quad (1.6)$$

in which $\tilde{\rho}^\pm, \tilde{u}_1^\pm$ and \tilde{E} are solution of (1.4). In the following discussion, we will also assume that the shift generated by the initial data satisfies

$$|\partial_{x'}^\beta(\delta_0^\pm(x') - \delta_*)| \leq C(1 + |x'|^2)^{-N}, \quad (1.7)$$

for any multi-index β and any positive integers N . Here, C is a constant depending only on β . This assumption implies that the shift $\delta_0(x')$ decays to δ_* almost exponentially. Again, this assumption can be reduced to the constraint on the initial perturbation. More precise construction of the background planar diffusion wave $(\bar{\rho}^\pm, \bar{u}^\pm)(x, t)$ and its properties will be given in Lemma 2.3 below.

Throughout this paper, we denote any generic constant by C . The usual Sobolev space is denoted by $W^{s,p}(\mathbf{R}^n)$, $s \in \mathbf{Z}_+$, $p \in [1, \infty]$ with the norm

$$\|f\|_{W^{s,p}} := \sum_{|\alpha|=0}^s \|\partial^\alpha f\|_{L^p},$$

where ∂^α used for ∂_x^α without confusion. In particular, $W^{s,2}(\mathbf{R}^n) = H^s(\mathbf{R}^n)$. Set

$$\begin{aligned} V^\pm(x, t) &= \rho^\pm(x, t) - \bar{\rho}^\pm(x, t), \\ U^\pm(x, t) &= (u_1^\pm(x, t) - \bar{u}_1^\pm(x, t), u_2^\pm(x, t), u_3^\pm(x, t)), \\ \nabla\varphi &= \nabla\phi - \bar{E} \quad (\text{Note } \operatorname{div}(\nabla\phi - \bar{E}) = 0), \\ K(x, t) &= V^+(x, t) - V^-(x, t), \end{aligned} \tag{1.8}$$

and also denote

$$\nu^\pm(x, 0) = \int_{-\infty}^{x_1} V^\pm(x_1, x', 0) dx_1, \quad \nu_t^\pm(x, 0) = \int_{-\infty}^{x_1} V_t^\pm(x_1, x', 0) dx_1. \tag{1.9}$$

Note that the time derivative on the initial data can be defined by the compatibility of the initial data through the equation (2.1).

Now we state the main results in this paper. Note that we consider only the spatial dimension $n = 3$ in this paper. However, we will still use notation n in the below theorem for convenience to extend our result to other high dimensional cases, since other higher dimensional cases can be considered similarly.

Theorem 1.1 *Let $(\bar{\rho}^\pm, \bar{u}^\pm)(x, t)$ in (1.6) be planar diffusion waves with a shift $\delta(x', t)$ constructed above. (See more precisely its properties in Lemma 2.3.) For $k \geq 4$, assume that the initial data $(\rho^\pm, u^\pm)(x, 0)$ satisfy the smallness assumption*

$$|\rho_+ - \rho_-| + \|(\nu^\pm, \nu_t^\pm)(\cdot, 0)\|_{L^2 \cap L^1} + \|(\rho^\pm - \bar{\rho}^\pm)(\cdot, 0)\|_{L^1} + \|(\rho^\pm - \bar{\rho}^\pm, u^\pm - \bar{u}^\pm)(\cdot, 0)\|_{H^k} \leq \epsilon_0,$$

where $\epsilon_0 > 0$ is a sufficiently small constant. Then

(i) (Global existence) There exist unique global classical solution $(\rho^\pm, u^\pm, \nabla\phi)$ to the system (1.1)-(1.2) that

$$V^\pm(t, x), U^\pm(t, x) \in C([0, \infty), H^k(\mathbf{R}^n)) \cap C^1((0, \infty), H^{k-1}(\mathbf{R}^n)), \quad \nabla\phi \in W^{k,6}(\mathbf{R}^n).$$

(ii) (L^p convergence) Moreover, for $|\gamma| \leq k-2$, $p \in [2, \infty]$, we have

$$\begin{aligned} \|\partial_x^\gamma V^\pm\|_{L^p} &\leq C\epsilon_0(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}, \\ \|\partial_x^\gamma U^\pm\|_{L^p} &\leq C\epsilon_0(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+2}{2}}. \end{aligned}$$

(iii) (Estimates on φ and $K = V^+ - V^-$) For $|\gamma| \leq k-2$,

$$\|\partial_x^\gamma K\|_{L^2} \leq C\epsilon_0(1+t)^{-\frac{5}{4}n-2-\frac{|\gamma|}{2}}, \quad \|\partial_x^\gamma \nabla\varphi\|_{L^6} \leq C\epsilon_0(1+t)^{-\frac{5}{4}n-2-\frac{|\gamma|}{2}},$$

and for $|\gamma| = k-1$,

$$\|\partial_x^\gamma K\|_{L^2} \leq C\epsilon_0(1+t)^{-\frac{5}{4}n-1-\frac{k}{2}}, \quad \|\partial_x^\gamma \nabla\varphi\|_{L^6} \leq C\epsilon_0(1+t)^{-\frac{5}{4}n-1-\frac{k}{2}}.$$

Remark 1.2 As noted in [20], in general, if the shift of the profile is not exactly captured, the decay rates for V^\pm and U^\pm should be $\frac{1}{2}$ lower than the one given in the above theorem even in one space dimensional case. Here, the reason that the above decay estimate holds

is that the shift due to the initial perturbation introduced above so that when we apply the Green function, the term corresponding to the initial data yields an extra $(1+t)^{-\frac{1}{2}}$ decay after taking the anti-derivative of the initial perturbation. Moreover, under the condition (1.7) on the initial shift, even though the anti-derivative of the perturbation can not be defined for all time as the shift function is not precisely defined, we know that δ_* is exactly the final shift when t tends to infinity of the profile because the initial perturbation will spread out eventually.

Remark 1.3 We note here that the L^2 decay rates of K is higher than that of V^\pm . However, we can only get the decay estimates on derivatives of K up to $(k-1)$ -th order.

Remark 1.4 Compared with [19, 27], our initial data are the small perturbation of the planar waves, instead of the constant states. In the meanwhile, here we can show the $L^p(p \in [2, +\infty])$ convergence rates of the planar waves of the three-dimensional bipolar Euler-Poisson equations. This improved the results in [7]. Moreover, here we only consider the case that the far fields of two particles' velocity in the x_1 -direction are same, see (1.5), namely, the switch-off case. However, we believe that the same results also hold for the switch-on case. Indeed, using the gap function with exponential decay in [7], we can show the similar results for the switch-on case.

The outline of the proof of the main theorems is as follows. First, we notice that the equations for V^\pm are coupled by $\nabla\varphi$, which is expressed by nonlocal Riesz potential $\nabla\varphi = \nabla\Delta^{-1}K$, with $K = V^+ - V^-$. So we need to have some good estimates on K before the estimate of V^+ and V^- . Luckily we note that K satisfies the damping "Klein-Gordon" type equations with an addition good term to perform the energy estimate. The estimates of K and $\nabla\varphi$ are given in Section 2, where the algebraic decay rates of K in the L^2 -norm are derived by some delicate energy methods, which will be used to obtain the $L^p(p \in [2, +\infty])$ -convergence rates of the solutions in the subsequent. Next, we use the frequency decomposition based energy method introduced in [20], which combines the approximation Green function and energy method, to prove global existence and L^p convergence results, see (i) and (ii) in Theorem 1.1. This method captures the low frequency component in the approximate Green function and avoids the singularity in the high frequency component. That is, we firstly show the precise algebraic decay estimate of V^\pm in the low frequency component, which dominant the decay of the perturbation, and then obtain the better decay rates of the high frequency component of V^\pm in the L^2 -norm by energy methods. For the high frequency component, one has an additional Poincaré-type inequality to close the energy estimate. Note that the lack of Poincaré inequality in the whole space is usually the essential difficulty in the energy estimate which is in contrast to the problem in a torus. This in some sense illustrates the essence of the Green function on the decay rate related to the frequency.

The rest of the paper is arranged as follows. In Section 2, we will reformulate the system around a planar diffusion wave defined in (1.3) and then state some known properties of this background diffusion wave. In Section 3, we will study the energy estimate of $K = V^+ - V^-$ and prove part (iii) in Theorem 1.1. The frequency decomposition based energy method will be carried out in Section 4 and 5, where in Section 4, we will study the approximate Green function and then the main L^p estimates on the low frequency component of V^\pm , and in

Section 5, we will study the L^2 energy estimates on the high frequency component of V^\pm . Finally, we will complete the proof to part (i) and (ii) of Theorem 1.1 in Section 6.

2 Preliminaries.

In this section, we will first derive the equations for the perturbation functions V^\pm and U^\pm defined in (1.8). Then we will recall some results on the background diffusion waves.

2.1 Reduced system.

We first derive the system for the perturbation of the nonlinear planar diffusion wave. Then, from (1.1) and (1.4), we have the equations for V^\pm that

$$V_t^\pm + (\bar{\rho}^\pm + V^\pm) \operatorname{div} U^\pm = R_{\rho^\pm} - (U^\pm \cdot \nabla)(\bar{\rho}^\pm + V^\pm) - V^\pm (\bar{u}_1^\pm)_{x_1} - \bar{u}_1^\pm V_{x_1}^\pm, \quad (2.1)$$

where

$$R_\rho^\pm = [-\bar{\rho}^\pm(x, t) \delta_t^\pm(x', t)]_{x_1}.$$

Similarly, the equations for U_1^\pm are

$$(U_1^\pm)_t + (\bar{\rho}^\pm + V^\pm)^{-1} [P(\bar{\rho}^\pm + V^\pm) - P(\bar{\rho}^\pm)]_{x_1} + U_1^\pm = \frac{P(\bar{\rho}^\pm)_{x_1} V^\pm}{\bar{\rho}^\pm(\bar{\rho}^\pm + V^\pm)} + R_{u_1^\pm} - R_1^\pm \pm \partial_{x_1} \varphi,$$

and for $i = 2, 3$,

$$(U_i^\pm)_t + (\bar{\rho}^\pm + V^\pm)^{-1} [P(\bar{\rho}^\pm + V^\pm) - P(\bar{\rho}^\pm)]_{x_i} + U_i^\pm = \frac{P(\bar{\rho}^\pm)_{x_i} V^\pm}{\bar{\rho}^\pm(\bar{\rho}^\pm + V^\pm)} - (\bar{\rho}^\pm)^{-1} P(\bar{\rho}^\pm)_{x_i} - R_i^\pm \pm \partial_{x_i} \varphi,$$

where

$$\begin{aligned} R_{u_1^\pm} &= -[\tilde{u}_1^\pm(x_1 + \delta(x', t), t) \delta_t(x', t)]_{x_1}, \\ R_1^\pm &= U^\pm \cdot \nabla(\bar{u}_1^\pm + U_1^\pm) + \bar{u}_1^\pm (U_1^\pm)_{x_1}, \\ R_i^\pm &= U^\pm \cdot \nabla U_i^\pm + \bar{u}_1^\pm (U_i^\pm)_{x_1}, \quad 2 \leq i \leq n. \end{aligned}$$

The equation for φ is simply

$$\Delta \varphi = V^+ - V^- = K,$$

it is directly that the perturbed electric field $\nabla \varphi$ can be expressed by the Riesz potential as a nonlocal term

$$\nabla \varphi = \nabla \Delta^{-1} K. \quad (2.2)$$

Then the system for the perturbation (V^\pm, U^\pm, φ) can be summarized as

$$\begin{cases} V_t^\pm + (\bar{\rho}^\pm + V^\pm) \operatorname{div} U^\pm = Q^\pm, \\ (U_i^\pm)_t + (\bar{\rho}^\pm + V^\pm)^{-1} (\mathcal{P}(V^\pm, \bar{\rho}^\pm) V^\pm)_{x_i} + U_i^\pm = H_i^\pm \pm \partial_{x_i} \varphi, \quad 1 \leq i \leq 3, \end{cases} \quad (2.3)$$

where $\mathcal{P}(V^\pm, \bar{\rho}^\pm) = \int_0^1 P'(\bar{\rho}^\pm + \theta V^\pm) d\theta$, and

$$\begin{aligned} Q^\pm &= R_{\rho^\pm} - (U^\pm \cdot \nabla)(\bar{\rho}^\pm + V^\pm) - V^\pm (\bar{u}_1^\pm)_{x_1} - (\bar{u}_1^\pm) V_{x_1}^\pm, \\ H_1^\pm &= R_{u_1^\pm} + \frac{P(\bar{\rho}^\pm)_{x_1} V^\pm}{\bar{\rho}^\pm(\bar{\rho}^\pm + V^\pm)} - R_1^\pm, \\ H_i^\pm &= -\frac{P(\bar{\rho}^\pm)_{x_i}}{\bar{\rho}^\pm} + \frac{P(\bar{\rho}^\pm)_{x_i} V^\pm}{\bar{\rho}^\pm(\bar{\rho}^\pm + V^\pm)} - R_i^\pm, \quad 2 \leq i \leq 3. \end{aligned}$$

Moreover, we can deduce the equation for $V^\pm(x, t)$ from (2.3) as

$$V_{tt}^\pm - \Delta[\mathcal{P}(V^\pm, \bar{\rho}^\pm)V^\pm] + V_t^\pm = \tilde{Q}(V^\pm, U^\pm, \bar{\rho}^\pm, \bar{u}_1^\pm) \mp \operatorname{div}[(\bar{\rho}^\pm + V^\pm)\nabla\varphi], \quad (2.4)$$

where

$$\begin{aligned} & \tilde{Q}(V^\pm, U^\pm, \bar{\rho}^\pm, \bar{u}_1^\pm) \\ &= [(R_{\rho^\pm})_t + R_{\rho^\pm}] - (1 + \partial_t)(V^\pm \bar{u}_1^\pm)_{x_1} - \operatorname{div}[(\bar{\rho}^\pm + V^\pm)_t U^\pm] - \operatorname{div}[(\bar{\rho}^\pm + V^\pm)H^\pm], \end{aligned}$$

with $H^\pm = (H_1^\pm, \dots, H_3^\pm)$. By linearizing (2.4) around $\bar{\rho}$, we have

$$\begin{aligned} & V_{tt}^\pm - \Delta(a^\pm(x, t)V^\pm) + V_t^\pm \\ &= \tilde{Q}(V^\pm, U^\pm, \bar{\rho}^\pm, \bar{u}_1^\pm) + \Delta(\mathcal{P}_1(\bar{\rho}, V)V^2) \mp \operatorname{div}[(\bar{\rho}^\pm + V^\pm)\nabla\varphi] \\ &=: F^\pm \mp \operatorname{div}[(\bar{\rho}^\pm + V^\pm)\nabla\varphi], \end{aligned} \quad (2.5)$$

where $a^\pm(x, t) = P'(\bar{\rho}^\pm)$ and

$$\mathcal{P}_1(\bar{\rho}^\pm, V^\pm) = \int_0^1 \left(\int_0^{\theta_1} P''(\bar{\rho}^\pm + \theta_2 V^\pm) d\theta_2 \right) d\theta_1.$$

Since

$$(R_{\rho^\pm})_t + R_{\rho^\pm} = (-\bar{\rho}_t^\pm(x, t)\delta_t(x', t))_{x_1},$$

direct calculation shows that $F^\pm = F(V^\pm, U^\pm, \bar{\rho}^\pm, \bar{u}^\pm)$ is in divergence form, that is,

$$F^\pm = \sum(F^{\pm,i})_{x_i} + \sum(F^{\pm,ij})_{x_i x_j}, \quad (2.6)$$

where, without confusion, we omit the \pm sign,

$$\begin{aligned} F^1 &= -\bar{\rho}_t \delta_t - (\bar{\rho} \bar{u}_1 \delta_t)_{x_1}, \quad F^i = -P(\bar{\rho})_{x_i}, \quad 2 \leq i \leq n, \\ F^{11} &= \bar{\rho}(2\bar{u}_1 U_1 + U_1^2) + V(\bar{u}_1 + U_1)^2 + \mathcal{P}_1(\bar{\rho}, V)V^2, \\ F^{1i} &= F^{i1} = 2[(\bar{\rho} + V)(\bar{u}_1 + U_1)U_i], \quad 2 \leq i \leq n, \\ F^{ij} &= (\bar{\rho} + V)U_i U_j + \delta_{ij}\mathcal{P}_1(\bar{\rho}, V)V^2, \quad 2 \leq i, j \leq n. \end{aligned}$$

Here δ_{ij} is the Kronecker symbol. On the other hand, by linearizing (2.3)₂ around \bar{u}^\pm , we have

$$U_t^\pm + (\bar{\rho}^\pm)^{-1}\nabla(a^\pm(x, t)V^\pm) + U^\pm = \bar{H}^\pm \pm \nabla\varphi, \quad (2.7)$$

where, again without confusion, we omit the \pm sign,

$$\begin{aligned} \bar{H}_1 &= R_u - \bar{\rho}^{-1}[\mathcal{P}_1(\bar{\rho}, V)V^2]_x - \frac{P(\bar{\rho}+V)_x V}{\bar{\rho}(\bar{\rho}+V)} - R_1, \\ \bar{H}_i &= -\frac{P(\bar{\rho})_{x_i}}{\bar{\rho}} - \frac{P(\bar{\rho}+V)_{x_i} V}{\bar{\rho}(\bar{\rho}+V)} - \bar{\rho}^{-1}[\mathcal{P}_1(\bar{\rho}, V)V^2]_{x_i} - R_i, \quad 2 \leq i \leq n. \end{aligned}$$

2.2 Background profile.

For later use, we include the following known estimates on the background planar wave, cf. [28]. By the definition of $W(x_1)$, we know for any integer N ,

$$\begin{aligned} \sup_{x_1>0} |W(x_1) - \rho_+| + \sup_{x_1<0} |W(x_1) - \rho_-| &\leq C|\rho_+ - \rho_-|(1+x_1^2)^{-N}, \\ |\partial_{x_1}^h W(x_1)| &\leq C|\rho_+ - \rho_-|(1+x_1^2)^{-N}, \quad (h>0). \end{aligned}$$

Recall that we have assumed in (1.7) for any multi-index β ,

$$|\partial_{x'}^\beta (\delta_0(x') - \delta_*)| \leq C(1+|x'|^2)^{-N}.$$

First, let us recall the results about the one-dimensional bipolar Euler-Poisson system (1.4).

Theorem 2.1 (see [4]) *Let $(\tilde{\rho}^\pm, \tilde{u}_1^\pm)(x_1, 0)$ be the initial data of one-dimensional bipolar Euler-Poisson system (1.4) and fix an integer $m \geq 2$. If there exists a small positive constant E_ρ such that the initial data $(\tilde{\rho}^\pm, \tilde{u}_1^\pm)(x_1, 0)$ satisfy*

$$\begin{aligned} |\rho_+ - \rho_-| + \left\| \int_{-\infty}^{x_1} (\tilde{\rho}^\pm(z, 0) - W(z)) dz \right\|_{L^2} \\ + \|\tilde{\rho}^\pm(\cdot, 0) - W(\cdot)\|_{H^{m+1}} + \|\tilde{u}_1^\pm(\cdot, 0) - \psi(\cdot, 0)\|_{H^{m+1}} \leq E_\rho, \end{aligned}$$

and

$$\int_{-\infty}^{x_1} (\tilde{\rho}^\pm(z, 0) - W(z)) dz, \tilde{u}^\pm + P(W)_{x_1} \in L^1(\mathbf{R}),$$

then (1.4) has global classical solution $(\tilde{\rho}^\pm, \tilde{u}^\pm, E)$ with

$$\begin{aligned} \|\partial^k (\tilde{\rho}^\pm - W)(x_1, t)\|_{L^p(\mathbf{R}_{x_1}^1)} &\leq CE_\rho(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k+1}{2}}, \\ \|\partial^k (\tilde{u}^\pm + P(W_{x_1}))(x_1, t)\|_{L^p(\mathbf{R}_{x_1}^1)} &\leq CE_\rho(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k+2}{2}} \end{aligned}$$

for any integer $k \leq m+1$ if $p=2$, and $k \leq m$ if $p=\infty$. Moreover, there exists a positive constant β such that

$$\|(\tilde{\rho}^+ - \tilde{\rho}^-, \tilde{E})\|_{H^m(\mathbf{R}_{x_1}^1)} \leq CE_\rho e^{-\beta t}.$$

Remark 2.2 Note that $(\tilde{\rho}^\pm, \tilde{u}_1^\pm)$ is an intermediate state we constructed to approximate the one-dimensional diffusion wave W . The assumptions on the initial data $(\tilde{\rho}^\pm, \tilde{u}_1^\pm)(x_1, 0)$, with $\tilde{\rho}^\pm(x_1, 0)$ connecting the two end states ρ_\pm , can be more regular than the assumptions on the initial date of the original problem (1.1).

Next, from the definition of the planar diffusion waves $(\bar{\rho}^\pm, \bar{u}^\pm)$ in (1.6), we can readily have

Lemma 2.3 *Under the assumptions in Theorem 2.1, the planar diffusion waves $(\bar{\rho}^\pm, \bar{u}^\pm)$ defined in (1.6) satisfy*

$$\begin{aligned} \sup_{x'} \|\partial^\alpha (\bar{\rho}_{x_1}^\pm, \bar{u}_1^\pm)(\cdot, x', t)\|_{L^2(\mathbf{R}_{x_1}^1)} &\leq CE_\rho(1+t)^{-\frac{1+|\alpha|}{2}-\frac{1}{4}}, \\ \|\partial^\alpha (\bar{\rho}_{x_1}^\pm, \bar{u}_1^\pm)(\cdot, t)\|_{L^\infty(\mathbf{R}^n)} &\leq CE_\rho(1+t)^{-\frac{1+|\alpha|}{2}} \end{aligned}$$

for any multi-index α with $|\alpha| \leq m - 1$, and

$$\|\bar{\rho}^+ - \bar{\rho}^-\|_{H^m(\mathbf{R}_{x_1}^1)} \leq CE_\rho e^{-\beta t}.$$

In addition, for $2 \leq i \leq n$,

$$\|\partial^\alpha(\bar{\rho}_{x_i}^\pm, (\bar{u}_1^\pm)_{x_i})(t)\|_{L^2(\mathbf{R}^n)} \leq CE_\rho e^{-t},$$

$$\|\partial^\alpha(\bar{\rho}_{x_i}^\pm, (\bar{u}_1^\pm)_{x_i})(t)\|_{L^\infty(\mathbf{R}^n)} \leq CE_\rho e^{-t}.$$

Note here again that we can increase the regularity of the assumptions on initial date of the one-dimensional problem to get sufficient estimates on the planar diffusion wave.

3 Estimates on $K = V^+ - V^-$.

In this section, we mainly give the estimate of $K = V^+ - V^-$. Recall the linearized equation (2.5) for V^\pm , we see that they are coupled by $\nabla\varphi$, which is expressed by the Riesz potential as in (2.2). i.e., $\nabla\varphi = \nabla\Delta^{-1}K$. So we need to have some good estimates on K , thus $\nabla\varphi$, before the estimate of V^+ and V^- . To begin with, we give a lemma on the relation of $\nabla\varphi$ and K .

Lemma 3.1 *If $K \in H^l(\mathbf{R}^n)$ for any integer $l > 1$, then $\nabla\varphi \in W^{l,6}(\mathbf{R}^n)$.*

Proof. Note that $\nabla\varphi$ be expressed by the Riesz potential

$$\nabla\varphi = \nabla\Delta^{-1}K = \mathcal{R} * K,$$

where $\hat{\mathcal{R}} = |2\pi\xi|^{-1}$ thus $\mathcal{R} = \frac{1}{|x|^{n-1}}$. Here $n = 3$ is the space dimension. Then by Hardy-Littlewood-Sobolev inequality [24] we have

$$\|\nabla\varphi\|_{L^6} = \|\mathcal{R} * K\|_{L^6} \leq C\|K\|_{L^2}, \quad (3.1)$$

and similarly, for any multi index $|\gamma| \leq k$,

$$\|\partial^\gamma\nabla\varphi\|_{L^6} = \|\mathcal{R} * \partial^\gamma K\|_{L^6} \leq C\|\partial^\gamma K\|_{L^2}, \quad (3.2)$$

that is, $\nabla\varphi \in W^{l,6}(\mathbf{R}^n)$ if $K \in H^l(\mathbf{R}^n)$.

Remark 3.2 *By Sobolev injection, this lemma automatically indicates*

$$\nabla\varphi \in L^\infty(\mathbf{R}^n).$$

Now we start to estimate K . The equation for K , from (2.5), is

$$K_{tt} - \Delta(a^+K) + K_t = F^+ - F^- - \operatorname{div}[(\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-)\nabla\varphi] + \Delta[(a^+ - a^-)V^-],$$

where F^\pm on the right hand side are defined in (2.6). This equation can also be written as

$$\begin{aligned} & K_{tt} - \Delta(a^+K) + K_t + (\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-)K \\ &= F^+ - F^- - \nabla(\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-) \cdot \nabla\varphi + \Delta[(a^+ - a^-)V^-], \end{aligned} \quad (3.3)$$

note here the last term on the left hand side is a good term, which ensures the closure of energy estimate for K . Note that the lack of such term in equation of V^\pm is the main difficulty in energy estimate thus we will use the frequency decomposition method introduced in [20].

To proceed, we first give the a priori assumption

$$\mathcal{M}(t) = \quad (3.4)$$

$$\max \left\{ \sup_{0 \leq s \leq t, |\alpha| \leq k-2, p \geq 2} (1+s)^{\frac{n}{2}(1-\frac{1}{p})+\frac{|\alpha|+1}{2}} \|\partial_x^\alpha V^\pm(\cdot, s)\|_{L^p}, \sup_{0 \leq s \leq t, |\alpha|=k, k-1} (1+s)^{\frac{n}{4}+\frac{|\alpha|+1}{2}} \|\partial_x^\alpha V^\pm(\cdot, s)\|_{L^2}, \right. \\ \left. \sup_{0 \leq s \leq t, |\alpha| \leq k-2, p \geq 2} (1+s)^{\frac{n}{2}(1-\frac{1}{p})+\frac{|\alpha|+2}{2}} \|\partial_x^\alpha U^\pm(\cdot, s)\|_{L^p}, \sup_{0 \leq s \leq t, |\alpha|=k, k-1} (1+s)^{\frac{n}{4}+\frac{k+1}{2}} \|\partial_x^\alpha U^\pm(\cdot, s)\|_{L^2} \right\}.$$

Under the assumption in Lemma 2.3 and the above a priori assumption, it is easy to check that for any multi-indices α and γ , the nonlinear terms in F^\pm satisfy

$$\begin{cases} \|\partial_y^\alpha F^i\|_{L^p(\mathbf{R}_y^n)} \leq CE_\rho e^{-s}, & |\alpha| \leq k, \\ \|\partial_y^\gamma F^{ij}\|_{L^p(\mathbf{R}_y^n)} \leq C\mathcal{M}^2(1+s)^{-(n+1+\frac{|\gamma|}{2})+\frac{n}{2p}}, & |\gamma| \leq k-2. \end{cases} \quad (3.5)$$

Here we should indicate that the above estimates hold for $p \geq 1$.

Now we perform energy estimates. Multiply equation (3.3) by K and integrate the resultant equation over \mathbf{R}^n , we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbf{R}^n} \left(\frac{1}{2} K^2 + KK_t \right) dx - \int_{\mathbf{R}^n} \Delta(a^+ K) K dx + \int_{\mathbf{R}^n} (\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-) K^2 dx - \int_{\mathbf{R}^n} K^2 dx \\ &= \int_{\mathbf{R}^n} K(F^+ - F^-) dx - \int_{\mathbf{R}^n} \nabla(\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-) \cdot \nabla \varphi K dx + \int_{\mathbf{R}^n} \Delta[(a^+ - a^-)V^-] K dx. \end{aligned}$$

Note

$$\int_{\mathbf{R}^n} -\Delta(a^+ K) K dx = \int_{\mathbf{R}^n} \nabla(a^+ K) \cdot \nabla K dx = \int_{\mathbf{R}^n} a^+ |\nabla K|^2 dx - \frac{1}{2} \int_{\mathbf{R}^n} (\Delta a^+) K^2 dx.$$

Moreover, using Cauchy-Schwarz's and Young's inequality, we have

$$\begin{aligned} \int_{\mathbf{R}^n} \nabla(\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-) \cdot \nabla \varphi K dx &\leq \|\nabla \varphi\|_{L^\infty} \|\nabla(\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-)\|_{L^2} \|K\|_{L^2} \\ &\leq C \|\nabla(\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-)\|_{L^2} \|K\|_{H^1}^2 \\ &\leq C(E_\rho + \mathcal{M}(t))(1+t)^{-\frac{5}{4}} \|K\|_{H^1}^2, \\ \int_{\mathbf{R}^n} \Delta[(a^+ - a^-)V^-] K dx &= \int_{\mathbf{R}^n} \nabla[(a^+ - a^-)V^-] \cdot \nabla K dx \\ &\leq C \|\nabla[(a^+ - a^-)V^-]\|_{L^2} \|\nabla K\|_{L^2} \\ &\leq CE_\rho e^{-\beta t} \mathcal{M}(t)((1+t)^{-\frac{n}{4}-\frac{1}{2}} + (1+t)^{-\frac{n}{4}-1}) \|\nabla K\|_{L^2}, \end{aligned}$$

since the Sobolev norm of $a^+ - a^-$ has same exponential decay as $\bar{\rho}^+ - \bar{\rho}^-$ in Lemma 2.3, and further

$$\int_{\mathbf{R}^n} K F^\pm dx \leq \varepsilon_0 \int_{\mathbf{R}^n} K^2 + C(\varepsilon_0) \int_{\mathbf{R}^n} (F^\pm)^2 \leq \varepsilon_0 \int_{\mathbf{R}^n} K^2 + C(\varepsilon_0)(E_\rho^2 + \mathcal{M}(t)^4)(1+t)^{-\frac{5}{2}n-4},$$

here we have used (3.5) in the above estimate. Combine above estimates and good decay properties of $\bar{\rho}^\pm$ in Lemma 2.3, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbf{R}^n} \left(\frac{1}{2} K^2 + K_t K \right) dx - \int_{\mathbf{R}^n} K_t^2 + \int_{\mathbf{R}^n} (K^2 dx + |\nabla K|^2) dx \\ & \leq C(E_\rho^2 + \mathcal{M}(t)^4)(1+t)^{-\frac{5}{2}n-4}. \end{aligned} \quad (3.6)$$

Here and in the subsequent we use the fact that $\bar{\rho}^\pm + V^\pm$ is strictly positive and bounded from below.

Next, multiply equation (3.3) by K_t and integrate the resultant equality over \mathbf{R}^n , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^n} K_t^2 dx - \int_{\mathbf{R}^n} \Delta(a^+ K) K_t dx + \int_{\mathbf{R}^n} K_t^2 dx + \int_{\mathbf{R}^n} (\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-) K K_t dx \\ & = \int_{\mathbf{R}^n} K_t (F^+ - F^-) dx - \int_{\mathbf{R}^n} \nabla(\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-) \cdot \nabla \varphi K_t dx + \int_{\mathbf{R}^n} \Delta[(a^+ - a^-) V^-] K_t dx. \end{aligned}$$

It is easy to compute

$$\begin{aligned} & \int_{\mathbf{R}^n} -\Delta(a^+ K) K_t dx = \int_{\mathbf{R}^n} \nabla(a^+ K) \cdot \nabla K_t dx \\ & = \int_{\mathbf{R}^n} a^+ \frac{1}{2} \partial_t |\nabla K|^2 dx + \int_{\mathbf{R}^n} \nabla a^+ \cdot (K \nabla K_t) dx \\ & = \int_{\mathbf{R}^n} a^+ \frac{1}{2} \partial_t |\nabla K|^2 dx + \int_{\mathbf{R}^n} \nabla a^+ \cdot \partial_t(K \nabla K) dx - \int_{\mathbf{R}^n} \nabla a^+ \cdot (K_t \nabla K) dx \\ & = \frac{d}{dt} \int_{\mathbf{R}^n} a^+ \frac{1}{2} |\nabla K|^2 dx - \int_{\mathbf{R}^n} \partial_t a^+ \frac{1}{2} |\nabla K|^2 dx - \frac{d}{dt} \int_{\mathbf{R}^n} \frac{1}{2} \Delta a^+ K^2 dx \\ & \quad + \int_{\mathbf{R}^n} \frac{1}{2} \partial_t \Delta a^+ K^2 dx - \int_{\mathbf{R}^n} \nabla a^+ \cdot (K_t \nabla K) dx, \end{aligned}$$

in which the last term on the right satisfies

$$\begin{aligned} \left| \int_{\mathbf{R}^n} \nabla a^+ \cdot (K_t \nabla K) dx \right| & \leq C \|\nabla a^+\|_{L^\infty} \|K_t\|_{L^2} \|\nabla K\|_{L^2} \\ & \leq \varepsilon_1 \|K_t\|_{L^2} + C(\varepsilon_1) E_\rho^2 (1+t)^{-\frac{3}{2}} \|\nabla K\|_{L^2}^2. \end{aligned}$$

Next, note also that $\bar{\rho}^\pm + V^\pm$ is strictly positive and bounded

$$\begin{aligned} & \int_{\mathbf{R}^n} (\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-) K K_t dx \\ & = \frac{d}{dt} \int_{\mathbf{R}^n} \frac{1}{2} (\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-) K^2 dx - \int_{\mathbf{R}^n} \frac{1}{2} \partial_t (\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-) K^2 dx. \end{aligned}$$

Also note

$$\begin{aligned} \int_{\mathbf{R}^n} \nabla(\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-) \cdot \nabla \varphi K_t & \leq C \|\nabla \varphi\|_{L^\infty} \|\nabla(\bar{\rho}^+ + \bar{\rho}^- + V^+ + V^-)\|_{L^2} \|K_t\|_{L^2} \\ & \leq \varepsilon_2 \|K_t\|_{L^2}^2 + C(\varepsilon_2) (E_\rho^2 + \mathcal{M}^2) (1+t)^{-\frac{3}{2}} \|K\|_{L^2}, \\ \int_{\mathbf{R}^n} \Delta[(a^+ - a^-) V^-] K_t dx & \leq \varepsilon_3 \|K_t\|_{L^2}^2 + C(\varepsilon_3) (E_\rho^2 + \mathcal{M}^2) t^{-\beta t}, \end{aligned}$$

and

$$\begin{aligned}\int_{\mathbf{R}^n} K_t F^\pm dx &\leq \varepsilon_0 \int K_t^2 dx + C(\varepsilon_0) \int_{\mathbf{R}^n} (F^\pm)^2 dx \\ &\leq \varepsilon_0 \int_{\mathbf{R}^n} K_t^2 dx + C(\varepsilon_0)(E_\rho^2 + \mathcal{M}^4)(1+t)^{-\frac{5}{2}n-4},\end{aligned}$$

then combine above estimates to get

$$\begin{aligned}&\frac{d}{dt} \int_{\mathbf{R}^n} (K_t^2 + K^2 + |\nabla K|^2) dx + \int_{\mathbf{R}^n} K_t^2 dx \\ &\leq CE_\rho \int_{\mathbf{R}^n} (K^2 + |\nabla K|^2) dx + C(E_\rho^2 + \mathcal{M}^4)(1+t)^{-\frac{5}{2}n-4}.\end{aligned}\tag{3.7}$$

Then multiply (3.6) by $\frac{1}{4}$ then add to (3.7), and if we assume that $E_\rho + \mathcal{M}$ is small enough, then

$$\frac{d}{dt} \int_{\mathbf{R}^n} (K^2 + |\nabla K|^2 + K_t^2) dx + \int_{\mathbf{R}^n} (K^2 + |\nabla K|^2 + K_t^2) \leq C(E_\rho^2 + \mathcal{M}^4)(1+t)^{-\frac{5}{2}n-4},$$

by Gronwall's inequality, we have

$$\int_{\mathbf{R}^n} K^2 + |\nabla K|^2 + K_t^2 \leq C(E_\rho^2 + \mathcal{M}^4)(1+t)^{-\frac{5}{2}n-4}.$$

then

$$\|K\|_{L^2} \leq C(E_\rho + \mathcal{M}^2)(1+t)^{-\frac{5}{4}n-2}.$$

For higher order derivatives (see also the estimate on high frequency part of V^\pm below), take ∂^γ on both side of equation (3.3), multiply by $\partial^\gamma K$ and $\partial^\gamma K_t$ and integrate, respectively, then combine as above estimates for lower oder derivative. Note that the last term on the right hand side of (3.3) $\Delta[(a^+ - a^-)V^-]$ has already second order derivative and the a priori assumption (3.4) has control of derivatives up to k -th order, so we can only carry out the computation for derivatives with order $|\gamma| \leq k-2$. Similar arguments as above, we have, for $|\gamma| \leq k-2$,

$$\|\partial^\gamma K\|_{L^2} \leq C(E_\rho + \mathcal{M}^2)(1+t)^{-\frac{5}{4}n-2-\frac{|\gamma|}{2}},\tag{3.8}$$

and for $|\gamma| = k-1$,

$$\|\partial_x^\gamma K\|_{L^2} \leq C\epsilon_0(1+t)^{-\frac{5}{4}n-1-\frac{k}{2}}.\tag{3.9}$$

Combine the estimates (3.1)-(3.2), (3.8)-(3.9) and using Lemma 3.1, we have the results of (iii) in Theorem 1.1.

4 Approximate Green Function and L^p Estimates on the Low Frequency Component.

In this section, we will give approximation Green function of the equation to V^\pm as in [20], which is used to get L^p estimates on the low frequency component of V^\pm . Recall the linearized equation (2.5),

$$V_{tt}^\pm - \Delta(a^\pm(x,t)V^\pm) + V_t^\pm = F^\pm \mp \operatorname{div}[(\bar{\rho}^\pm + V^\pm)\nabla\varphi].\tag{4.1}$$

We slightly abuse notations by dropping ‘ \pm ’ sign without confusion in the following estimates. Note the main difference of (4.1) to the linearized equation in [20] is that we need to consider the coupling term ‘ $\mp \operatorname{div}[(\bar{\rho}^\pm + V^\pm)\nabla\varphi]$ ’ in the present setting.

4.1 Approximate Green Function.

In this subsection, we study the approximate Green function for (4.1). For convenience of readers, we briefly repeat the construction of the approximate Green function in below.

Let $G(x, t; y, s)$ be the approximate Green function for the homogeneous part of (4.1) which meets the basic requirement

$$G(x, t; y, t) = 0, \quad G_t(x, t; y, t) = \delta(x - y),$$

where δ is the Dirac function. Multiplying (4.1) whose variables are now changed to (y, s) by G and integrating with respect to y and s over the region $\mathbf{R}^n \times [0, t]$ to get (note here and below we dropped the notation ‘ \pm ’)

$$\begin{aligned} V(x, t) = & \int_{\mathbf{R}^n} G_s(x, t; y, 0)V(y, 0)dy - \int_{\mathbf{R}^n} G(x, t; y, 0)(V + V_s)(y, 0)dy \\ & + \int_0^t \int_{\mathbf{R}^n} (G_{ss} - a^+ \Delta_y G - G_s)(x, t; y, s)V(y, s)dyds \\ & - \int_0^t \int_{\mathbf{R}^n} G(x, t; y, s)F(y, s)dyds \\ & + \int_0^t \int_{\mathbf{R}^n} G(x, t; y, s)\operatorname{div}[(\bar{\rho} + V)\nabla\varphi](y, s)dyds. \end{aligned} \tag{4.2}$$

If $a(y, s)$ is a constant and G is the Green function of the homogeneous part of (4.1), then the third integral in above is zero. However, when $a(y, s)$ is not a constant, it is difficult to give an explicit expression of the Green function. Therefore, we will use the approximate Green function constructed in [28, 20]. The idea is to first consider the linear partial differential equation

$$\partial_{tt}V - \mu\Delta V + V_t = 0, \tag{4.3}$$

where μ is a bounded parameter with $C_0 < \mu < C_1$, and denote its Green function by $G^\sharp(\mu; x, t)$, whose Fourier transform

$$\hat{G}^\sharp(\mu; \xi, t) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-},$$

where

$$\lambda_\pm(\xi) \equiv \frac{1}{2}(-1 \pm \sqrt{1 - 4\mu|\xi|^2}).$$

Denote $\hat{G}^\sharp = \hat{E}^+ + \hat{E}^-$ with

$$\hat{E}^+ = \eta_0 e^{\lambda_+ t}, \quad \hat{E}^- = \eta_0 e^{\lambda_- t}, \quad \eta_0 = (\lambda_+ - \lambda_-)^{-1}.$$

The approximate Green function is defined by

$$G(x, t; y, s) = G^\sharp(a(y, \sigma(t, s)); x - y, t - s), \tag{4.4}$$

with $a(y, \sigma(t, s)) = P'(\bar{\rho}(y, \sigma(t, s)))$, and the function $\sigma(t, s)$ is chosen such that $\sigma(t, s) \in C^3([2, \infty] \times [0, \infty])$,

$$\sigma(t, s) = \begin{cases} s, & s > t/2, \\ t/2, & s \leq t/2 - 1, \end{cases}$$

and

$$\sum_{1 \leq l_1 + l_2 \leq 3} |\partial_t^{l_1} \partial_s^{l_2} \sigma(t, s)| \leq C, \quad s \in (t/2 - 1, t/2).$$

Notice that $\sigma^{-1}(t, s) \leq C(1+t)^{-1}$ for $t > 2$ so that we have by Lemma 2.1

$$(1+t)|\partial_s a(y, \sigma(t, s))| + (1+t)^2 |\partial_s^2 a(y, \sigma(t, s))| \leq C E_\rho, \quad (4.5)$$

where E_ρ is defined in Lemma 2.1.

Notice that the decay of the derivatives of the function $a(y, \sigma(t, s))$ with respect to time will be used in the following analysis. Recall that the approximate Green function defined in (4.4) is not symmetric with respect to the variables (x, t) and (y, s) . However, straightforward calculation gives their relations as

$$\partial_{x_i} G = -\partial_{y_i} G + \partial_a(G^\sharp) a_{x_i}, \quad \partial_t G = -\partial_s G + \partial_a(G^\sharp) (a_s + a_t). \quad (4.6)$$

Denote the low frequency component in the approximate Green function $G(x, t; y, s)$ by

$$G_L(x, t; y, s) = \chi(D_x) G(x, t; y, s),$$

where $\chi(D_x)$, $D_x = \frac{1}{\sqrt{-1}} \partial_x = \frac{1}{\sqrt{-1}} (\partial_{x_1}, \dots, \partial_{x_n})$, is the pseudo-differential operator with symbol $\chi(\xi)$ as a smooth cut-off function satisfying

$$\chi(\xi) = \begin{cases} 1, & |\xi| < \varepsilon, \\ 0, & |\xi| > 2\varepsilon, \end{cases}$$

for some chosen constant ε in $(0, \varepsilon_0)$ with $\varepsilon_0 = \frac{1}{2} \min \left\{ 1, \sqrt{\frac{1}{4C_1}} \right\}$, C_1 is the upper bound of μ in (4.3). Moreover, we have

$$\begin{aligned} G_L(x, t; y, s) &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \chi(\xi) e^{\sqrt{-1}(x-y)\xi} \hat{G}^\sharp(a(y, \sigma(t, s)), \xi, t-s) d\xi \\ &= G_L^\sharp(a(y, \sigma(t, s)); x-y, t-s). \end{aligned}$$

Therefore, it is direct to have the following proposition [20].

Proposition 4.1 *For $q \in [1, \infty]$ and any indices h, l, α and β , we have*

$$\begin{aligned} \sup_y \|\partial_x^\alpha \partial_y^\beta \partial_s^l \partial_t^h G_L(\cdot, t; y, s)\|_{L^q(\mathbf{R}_x^n)} &\leq C(1+t-s)^{-\frac{n}{2}(1-\frac{1}{q}) - \frac{2\min(l+h, 1)+|\alpha|+|\beta|}{2}}, \\ \sup_x \|\partial_x^\alpha \partial_y^\beta \partial_s^l \partial_t^h G_L(x, t; \cdot, s)\|_{L^q(\mathbf{R}_y^n)} &\leq C(1+t-s)^{-\frac{n}{2}(1-\frac{1}{q}) - \frac{2\min(l+h, 1)+|\alpha|+|\beta|}{2}}. \end{aligned}$$

4.2 L^p Estimates on the Low Frequency Component.

In this subsection, we will establish the L^p estimates on the low frequency component by using the approximate Green function. Assume that $|\alpha| \leq k$ in this section. To derive the L^p estimates for the low frequency part, recall (4.2), and set

$$\begin{aligned} I_1^\alpha &= \chi(D_x) \int_{\mathbf{R}^n} \partial_x^\alpha G_s(x, t; y, 0) V(y, 0) dy = \int_{\mathbf{R}^n} \partial_x^\alpha (G_L)_s(x, t; y, 0) V(y, 0) dy, \\ I_2^\alpha &= - \int_{\mathbf{R}^n} \partial_x^\alpha G_L(x, t; y, 0) (V + V_s)(y, 0) dy, \\ I_3^\alpha &= \int_0^t \int_{\mathbf{R}^n} \partial_x^\alpha R_{G_L}(x, t; y, s) V(y, s) dy ds, \\ I_4^\alpha &= - \int_0^t \int_{\mathbf{R}^n} \partial_x^\alpha G_L(x, t; y, s) F(y, s) dy ds, \\ I_5^\alpha &= \int_0^t \int_{\mathbf{R}^n} \partial_x^\alpha G_L(x, t; y, s) \operatorname{div}[(\bar{\rho} + V) \nabla \varphi](y, s) dy ds, \end{aligned}$$

where

$$R_G \equiv G_{ss}(x, t; y, s) - a^+(y, s) \Delta_y G(x, t; y, s) - G_s(x, t; y, s),$$

and

$$\chi(D_x) R_G = R_{G_L}.$$

Since

$$(G_{tt}^\sharp - a \Delta G^\sharp + G_t^\sharp)(a(y, s); x - y, t - s) = 0,$$

we have

$$\begin{aligned} &R_{G_L}(x, t; y, s) \\ &= \left[G_{L;0,0}^\sharp(a(y, s); x - y, t - s) a_s(y, \sigma)^2 - 2G_{L;0,n+1}^\sharp(a(y, s); x - y, t - s) a_s(y, \sigma) \right. \\ &\quad + G_{L;0}^\sharp(a(y, s); x - y, t - s) a_{ss}(y, \sigma) - G_{L;0}^\sharp(a(y, s); x - y, t - s) a_s(y, \sigma) \\ &\quad + a(y, s) \left(\sum_{i=1}^n [G_{L;0,i}^\sharp(a(y, s); x - y, t - s) a_{yi}(y, \sigma) \right. \\ &\quad \left. - G_{L;0,0}^\sharp(a(y, s); x - y, t - s) ((a)_{yi}^2)(y, \sigma)] + G_{L;0}^\sharp(a(y, s); x - y, t - s) \Delta_y a(y, \sigma) \right) \Big] \\ &\quad + [(a(y, \sigma) - a(y, s)) \Delta G_L^\sharp(a(y, s); x - y, t - s)] \\ &=: R_{G_L}^1 + R_{G_L}^2. \end{aligned}$$

Here $R_{G_L}^i$, $i = 1, 2$, is the corresponding term in the above summation in the above equation.

To denote the derivatives, we use the notations $G_{L;0}^\sharp(a; x, t) = \partial_a G_L^\sharp(a; x, t)$, $G_{L;i}^\sharp(a; x, t) = \partial_{x_i} G_L^\sharp(a; x, t)$, $G_{L;n+1}^\sharp(a; x, t) = \partial_t G_L^\sharp(a; x, t)$, $G_{L;0,i}^\sharp(a; x, t) = \partial_a \partial_{x_i} G_L^\sharp(a; x, t)$, and $G_{L;0,n+1}^\sharp(a; x, t) = \partial_a \partial_t G_L^\sharp(a; x, t)$ etc..

Then, set $X_L(x, t) = \chi(D_x) X(x, t)$, from above notations we have

$$\partial_x^\alpha X_L(x, t) = I_1^\alpha + I_2^\alpha + I_3^\alpha + I_4^\alpha + I_5^\alpha.$$

We will estimate the right hand side of above term by term. The terms from I_1^α to I_4^α are similar to the estimates in [20]: by Proposition 4.1, it is straightforward to obtain

$$\|I_1^\alpha\|_{L^p(\mathbf{R}_x^n)} \leq C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+2}{2}} \|V_0\|_{L_1}.$$

For I_2^α , set

$$\tilde{\nu}_0(y) = \nu_t(y, 0) + \nu(y, 0),$$

where ν (with \pm omitted) is defined in (1.9). Then

$$|I_2^\alpha| = \left| \int_{\mathbf{R}^n} \partial_{y_1} \partial_x^\alpha G_L(x, t; y, 0) \tilde{\nu}_0(y) dy \right|.$$

Also by using Propositions 4.1, we have

$$\|I_2^\alpha\|_{L^p(\mathbf{R}_x^n)} \leq C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}} \|\tilde{\nu}_0\|_{L_1}.$$

We now turn to estimate the term I_3^α which is the error coming from the approximate Green function. For illustration, we only consider

$$J_1^\alpha = \int_0^t \int_{\mathbf{R}^n} \partial_x^\alpha G_{L;0}^\sharp(a^+(y, \sigma), x-y, t-s) a_s^+(y, \sigma) V(y, s) dy ds,$$

and

$$J_2^\alpha = \int_0^t \int_{\mathbf{R}^n} \partial_x^\alpha R_{G_L}^2(x, t; y, s) V(y, s) dy ds,$$

because the other terms in I_3^α can be estimated similarly. Note that (4.5) gives

$$|a_s^+(y, \sigma)| \leq CE_\rho(1+t)^{-1},$$

then we have, for $|\gamma| \leq k-2$,

$$\begin{aligned} \|J_1^\gamma\|_{L^p(\mathbf{R}_x^n)} &\leq \int_0^t \left\| \int_{\mathbf{R}^n} \partial_x^\gamma G_{L;0}^\sharp(a(y, \sigma), x-y, t-s) a_s(y, \sigma) V(y, s) dy \right\|_{L^p(\mathbf{R}_x^n)} ds \\ &\leq CE_\rho \mathcal{M} \left[\int_0^{t/2} (1+t-s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|}{2}} (1+t)^{-1} (1+s)^{-\frac{n}{2}(1-\frac{1}{p})-1/2} ds \right. \\ &\quad \left. + \int_{t/2}^t (1+t-s)^{-\frac{n}{2}(1-\frac{1}{p})} (1+t)^{-1} (1+s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}} ds \right] \\ &\leq CE_\rho \mathcal{M} (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}, \end{aligned}$$

and for $|\gamma| = k-1$ and k ,

$$\begin{aligned} \|J_1^\gamma\|_{L^p(\mathbf{R}_x^n)} &\leq CE_\rho \mathcal{M} \int_0^{t/2} (1+t-s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|}{2}} (1+t)^{-1} (1+s)^{-\frac{n}{2}(1-\frac{1}{p})-1/2} ds \\ &\quad + CE_\rho \mathcal{M} \int_{t/2}^t (1+t-s)^{-\frac{|\gamma|+2-k}{2}} (1+t)^{-1} (1+s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{k-2+1}{2}} ds \\ &\leq CE_\rho \mathcal{M} (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}. \end{aligned}$$

For J_2^α , since

$$|a(y, s) - a(y, \sigma)| \leq \int_s^\sigma |a_\tau(y, \tau)| d\tau \leq \begin{cases} CE_\rho \Theta(t, s), & s < t/2, \\ 0, & s \geq t/2, \end{cases}$$

where

$$\Theta(t, s) = (1 + t - s)(1 + t)^{-1+1/h}(1 + s)^{-1/h},$$

and h can be any positive integer. By using Proposition 4.1, we have

$$\|J_2^\alpha\|_{L^p(\mathbf{R}_x^n)} \leq CE_\rho \mathcal{M} \int_0^{t/2} (1 + t - s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+2}{2}} \Theta(t, s) (1 + s)^{-1/2} ds.$$

By noticing that

$$\begin{aligned} \int_0^{t/2} (1 + t)^{-1+1/h} (1 + s)^{-1/h} (1 + s)^{-1/2} ds &= (1 + t)^{-1+1/h} (1 + s)^{\frac{1}{2}-\frac{1}{h}} \Big|_0^{t/2} \\ &\leq C(1 + t)^{-1+1/h} (1 + t)^{\frac{1}{2}-\frac{1}{h}} = C(1 + t)^{-\frac{1}{2}}, \end{aligned}$$

we obtain

$$\|J_2^\alpha\|_{L^p(\mathbf{R}_x^n)} \leq CE_\rho \mathcal{M} (1 + t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}.$$

Thus, combine the above estimate to have

$$\|I_3^\alpha\|_{L^p(\mathbf{R}_x^n)} \leq CE_\rho \mathcal{M} (1 + t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}.$$

Next for I_4^α , recall that F (with \pm omitted) satisfies (3.5) under the a priori assumption (3.4), then for $|\gamma| \leq k - 2$,

$$\begin{aligned} \|I_4^\gamma\|_{L^p(\mathbf{R}_x^n)} &\leq \int_0^t \left\| \int_{\mathbf{R}^n} \partial_x^\gamma G_L F dy \right\|_{L^p(\mathbf{R}_x^n)} ds \\ &= \int_0^{\frac{t}{2}} \left\| \int_{\mathbf{R}^n} \left(- \sum \partial_x^\gamma \partial_{y_i} G_L F^i + \sum \partial_x^\gamma \partial_{y_i y_j} G_L F^{ij} \right) dy \right\|_{L^p(\mathbf{R}_x^n)} ds \\ &\quad + \int_{\frac{t}{2}}^t \left\| \int_{\mathbf{R}^n} \partial_x^\gamma G_L F dy \right\|_{L^p(\mathbf{R}_x^n)} ds \\ &\leq CE_\rho \int_0^{\frac{t}{2}} (1 + t - s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}} e^{-s} ds + CE_\rho \int_{\frac{t}{2}}^t (1 + t - s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}} e^{-s} ds \\ &\quad + C\mathcal{M}^2 \int_0^{\frac{t}{2}} (1 + t - s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+2}{2}} (1 + s)^{-(n+1)+\frac{n}{2}} ds \\ &\quad + C\mathcal{M}^2 \int_{\frac{t}{2}}^t (1 + t - s)^{-1} (1 + s)^{-(n+1+\frac{|\gamma|}{2})+\frac{n}{2p}} ds \\ &\leq C(E_\rho + \mathcal{M}^2) (1 + t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}. \end{aligned}$$

The cases when $|\gamma| = k - 1$ and k can be estimated similarly, and the only difference is the estimation on the terms like

$$\int_{\frac{t}{2}}^t \left\| \int_{\mathbf{R}^n} \partial_x^\gamma \partial_{y_i y_j} G_L F^{ij} dy \right\|_{L^p(\mathbf{R}_x^n)} ds, \quad |\gamma| \leq k - 2.$$

On the other hand, these terms can be estimated by replacing the derivatives of G_L w.r.t. x to the derivatives of G_L w.r.t. y using (4.6). Then by using integration by parts $k - 2$ times

to transfer the derivatives on G_L to F^{ij} , we have by (4.8) and Proposition 4.1 that

$$\begin{aligned} \int_{\frac{t}{2}}^t \left\| \int_{\mathbf{R}^n} \partial_x^\gamma G_L \partial_{y_i y_j} F^{ij} dy \right\|_{L^p(\mathbf{R}_x^n)} ds &\leq C\mathcal{M}^2 \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{|\gamma|+2-(k-2)}{2}} (1+s)^{-(n+1+\frac{k-2}{2})+\frac{n}{2p}} ds \\ &\leq C\mathcal{M}^2 (1+t)^{-(n+1+\frac{k-2}{2})+\frac{n}{2p}} \\ &\leq C\mathcal{M}^2 (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}. \end{aligned}$$

Therefore, we have the L^p estimate on I_4^α as

$$\|I_4^\alpha\|_{L^p(\mathbf{R}_x^n)} \leq C\mathcal{M}^2 (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}.$$

For I_5^α , we write

$$\begin{aligned} I_5^\alpha &= \int_0^{\frac{t}{2}} \int_{\mathbf{R}^n} \partial_x^\alpha \nabla_y G_L(x, t; y, s) \cdot [(\bar{\rho} + V) \nabla \varphi](y, s) dy ds \\ &\quad + \int_{\frac{t}{2}}^t \int_{\mathbf{R}^n} \partial_x^\alpha G_L(x, t; y, s) \operatorname{div}[(\bar{\rho} + V) \nabla \varphi](y, s) dy ds \\ &=: I_{5,1}^\alpha + I_{5,2}^\alpha, \end{aligned}$$

in which the first term satisfies

$$I_{5,1}^\alpha \leq C \int_0^{\frac{t}{2}} \|\partial_x^\alpha \nabla_y G_L\|_{L^P} \|(\bar{\rho} + V) \nabla \varphi\|_{L^1} ds.$$

To estimate

$$\|(\bar{\rho} + V) \nabla \varphi\|_{L^1} \leq C \|\bar{\rho} + V\|_{L^{\frac{6}{5}}} \|\nabla \varphi\|_{L^6},$$

we note that the $L^{\frac{6}{5}}$ norm of $\bar{\rho} + V (= \rho, positive)$ can be controlled by its L^1 and L^2 norms by interpolation, and also note the L^1 norm of ρ (with \pm omitted) is conserved because the conservation of mass, L^2 norm can be bounded by the a priori assumption, then we have

$$\|I_{5,1}^\alpha\|_{L^P} \leq C\mathcal{M}^2 (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}.$$

For the term $I_{5,2}^\alpha$, we estimate

$$I_{5,2}^\alpha \leq C \int_{\frac{t}{2}}^t \|\partial_x^\alpha G_L\|_{L^1} \|\operatorname{div}[(\bar{\rho} + V) \nabla \varphi]\|_{L^p} ds.$$

Note that both $\operatorname{div} \nabla \varphi = K$ and $\nabla \varphi$ are in L^∞ thus have good decay properties, then the above term decays faster than that of $I_{5,1}^\alpha$, then we have

$$\|I_5^\alpha\|_{L^P} \leq C\mathcal{M}^2 (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}.$$

In summary, by combining all above estimates, we have the estimates on the low frequency component of V^\pm in the following theorem.

THEOREM 4.2. *For $|\alpha| \leq k$, we have,*

$$\|\partial^\alpha V_L^\pm(t)\|_{L^p} \leq C(E_0 + \mathcal{M}^2)(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}},$$

where $E_0 = \max\{\|V_0^\pm\|_{L^1}, \|\tilde{V}_0^\pm\|_{L^1}, \|(V_0^\pm, U_0^\pm)\|_{H^k}, \|V_t^\pm(0)\|_{H^{k-1}}, E_\rho\}$.

As an immediate consequence, we have the L^2 estimate on the derivatives of order higher than the k -th for the low frequency component because

$$\|\partial_{x_i} \partial^\alpha V_L^\pm(t)\|_{L^2} = \|\xi_i \xi^\alpha \chi(\xi) \hat{V}^\pm\|_{L^2} \leq \varepsilon \|\xi^\alpha \chi(\xi) \hat{V}^\pm\|_{L^2} = \varepsilon \|\partial^\alpha V_L^\pm(t)\|_{L^2}.$$

Thus, we have the following corollary.

Corollary 4.2 *For any $|\gamma| > k$, we have*

$$\|\partial^\gamma V_L^\pm(t)\|_{L^2} \leq C(E_0 + \mathcal{M}^2) \varepsilon (1+t)^{-\frac{n}{2}(1-\frac{1}{2}) - \frac{k+1}{2}}.$$

5 Estimates on the High Frequency Component.

In this section, we will carry out the energy estimates on the high frequency component. Recall the linearized equation (2.5), set $\tilde{\chi}(\xi) = 1 - \chi(\xi)$ and $V_H^\pm(x, t) = \tilde{\chi}(D_x) V^\pm(x, t)$. By taking $\tilde{\chi}(D_x)$ on both sides of (2.5) and integrating its product with V_H^\pm and $(V_H^\pm)_t$ over \mathbf{R}^n respectively, we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{R}^n} V_H^\pm (V_H^\pm)_t dx - \int_{\mathbf{R}^n} ((V_H^\pm)_t)^2 dx - \int_{\mathbf{R}^n} V_H^\pm \Delta \tilde{\chi}(aV^\pm) dx + \frac{d}{dt} \int_{\mathbf{R}^n} \frac{1}{2} (V_H^\pm)^2 dx \\ = \int_{\mathbf{R}^n} V_H^\pm \tilde{\chi} F^\pm dx \mp \int_{\mathbf{R}^n} (V_H^\pm)_t \tilde{\chi} \operatorname{div}[(\bar{\rho}^\pm + V^\pm) \nabla \varphi], \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{R}^n} \frac{1}{2} ((V_H^\pm)_t)^2 dx - \int_{\mathbf{R}^n} (V_H^\pm)_t \Delta \tilde{\chi}(aV^\pm) dx + \int_{\mathbf{R}^n} ((V_H^\pm)_t)^2 dx \\ = \int_{\mathbf{R}^n} (V_H^\pm)_t \tilde{\chi} F^\pm dx \mp \int_{\mathbf{R}^n} V_H^\pm \tilde{\chi} \operatorname{div}[(\bar{\rho}^\pm + V^\pm) \nabla \varphi]. \end{aligned} \quad (5.2)$$

Again, we slightly abuse notations by dropping the \pm sign without confusion in the following estimates. First for the third term on the left hand side of (5.1) as follows. That is,

$$- \int_{\mathbf{R}^n} V_H \Delta \tilde{\chi}(aV) dx = \int_{\mathbf{R}^n} a |\nabla V_H|^2 dx - \int_{\mathbf{R}^n} V_H \nabla [\nabla \tilde{\chi}, a] V dx,$$

where $[A, B] = A \circ B - B \circ A$ denotes the commutator. Since

$$(1+t)^{1/2} \|\nabla a\|_{L^\infty} + (1+t) \|\Delta a\|_{L^\infty} \leq C E_\rho,$$

where E_ρ is defined in Theorem 2.1. It is straightforward to show that

$$\int_{\mathbf{R}^n} |\nabla [\nabla \tilde{\chi}, a] V|^2 dx \leq C E_\rho^2 \mathcal{M}^2 (1+t)^{-\frac{n}{2}-3},$$

where \mathcal{M} is defined in (3.4). Thus

$$| \int_{\mathbf{R}^n} V_H \nabla [\nabla \tilde{\chi}, a] V dx | \leq \eta \int_{\mathbf{R}^n} |V_H|^2 dx + C E_\rho^2 \mathcal{M}^2 (1+t)^{-\frac{n}{2}-3}.$$

We now turn to estimate the second term on the left hand side in (5.2). That is,

$$\begin{aligned} -\int_{\mathbf{R}^n} (V_H)_t \Delta \tilde{\chi}(aV) dx &= \int_{\mathbf{R}^n} (\nabla V_H)_t a \nabla \tilde{\chi}(V_H) dx + \int_{\mathbf{R}^n} (\nabla V_H)_t [\nabla \tilde{\chi}, a] V dx \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^n} a |V_H|^2 dx - \frac{1}{2} \int_{\mathbf{R}^n} a_t |V_H|^2 dx - \int_{\mathbf{R}^n} (V_H)_t \nabla [\nabla \tilde{\chi}, a] V dx, \end{aligned}$$

in which, we have

$$\left| \int_{\mathbf{R}^n} a_t |\nabla(V_H)|^2 dx \right| \leq C E_\rho^2 (1+t)^{-2} \int_{\mathbf{R}^n} |\nabla(V_H)|^2 dx,$$

and

$$\left| \int_{\mathbf{R}^n} (V_H)_t \nabla [\nabla \tilde{\chi}, a] V dx \right| \leq \eta \int_{\mathbf{R}^n} |(V_H)_t|^2 dx + C E_\rho^2 \mathcal{M}^2 (1+t)^{-\frac{n}{2}-3}.$$

For $\int_{\mathbf{R}^n} V_H \tilde{\chi} F dx$ and $\int_{\mathbf{R}^n} (V_H)_t \tilde{\chi} F dx$ on the right hand side of (5.1) and (5.2), by using Lemma 2.3 and the definition of \mathcal{M} in (3.4), we have

$$\begin{aligned} \|F^i\|_{L^2} &\leq C E_\rho e^{-s}, \\ \|\partial^\gamma F^{ij}\|_{L^2} &\leq C \mathcal{M}^2 (1+t)^{-(n+1+\frac{|\gamma|}{2})+\frac{n}{4}}, \quad |\gamma| \leq k-2, \end{aligned}$$

where F and F^i, F^{ij} defined in (2.6). Further, it is straightforward to check that

$$\left| \int_{\mathbf{R}^n} V_H \tilde{\chi} F dx \right| \leq \eta \int_{\mathbf{R}^n} |V_H|^2 dx + C(\eta)(E_\rho^2 + \mathcal{M}^4)(e^{-t} + (1+t)^{-2(n+1)+\frac{n}{2}}),$$

and

$$\left| \int_{\mathbf{R}^n} (V_H)_t \tilde{\chi} F dx \right| \leq \eta \int_{\mathbf{R}^n} |(V_H)_t|^2 dx + C(\eta)(E_\rho^2 + \mathcal{M}^4)(e^{-t} + (1+t)^{-2(n+1)+\frac{n}{2}}).$$

For the term $\int_{\mathbf{R}^n} (V_H)_t \tilde{\chi} \operatorname{div}[(\bar{\rho} + V) \nabla \varphi]$, we have

$$\begin{aligned} &\int_{\mathbf{R}^n} (V_H)_t \tilde{\chi} \operatorname{div}[(\bar{\rho} + V) \nabla \varphi] dx \\ &\leq \varepsilon \int_{\mathbf{R}^n} (V_H)_t^2 dx + C(\varepsilon) \int_{\mathbf{R}^n} (\operatorname{div}[(\bar{\rho} + V) \nabla \varphi])^2 dx \\ &= \varepsilon \int_{\mathbf{R}^n} (V_H)_t^2 dx + C(\varepsilon) \int_{\mathbf{R}^n} [\nabla(\bar{\rho} + V) \cdot \nabla \varphi + (\bar{\rho} + V) K]^2 dx \\ &\leq \varepsilon \int_{\mathbf{R}^n} (V_H)_t^2 dx + 2C(\varepsilon) \left(\|\nabla(\bar{\rho} + V)\|_{L^2}^2 \|\nabla \varphi\|_{L^\infty}^2 + \|\bar{\rho} + V\|_{L^\infty}^2 \|K\|_{L^2}^2 \right) \\ &\leq \varepsilon \int_{\mathbf{R}^n} (V_H)_t^2 dx + C(\varepsilon)(E_\rho^2 + \mathcal{M}^2)(1+t)^{-\frac{3}{2}n-|\alpha|-2}. \end{aligned}$$

The term $\int_{\mathbf{R}^n} V_H \tilde{\chi} \operatorname{div}[(\bar{\rho} + V) \nabla \varphi] dx$ can be estimated similarly.

To close the energy estimate, one needs the following important fact about the high frequency part:

$$\int_{\mathbf{R}^n} |\nabla V_H|^2 dx \geq \varepsilon \int_{\mathbf{R}^n} |V_H|^2 dx.$$

This is a Poincaré type inequality which holds only for the high frequency part in the whole space. By integrating (5.1) and (5.2) over $[0, t]$ and multiplying (5.1) by some suitably chosen constant $0 < \lambda < 1$, when η is small, the combination of above estimates give

$$\begin{aligned} & \int_{\mathbf{R}^n} (|V_H|^2 + |(V_H)_t|^2 + |\nabla V_H|^2)(t) dx + \mu \int_0^t \int_{\mathbf{R}^n} (|V_H|^2 + |(V_H)_s|^2 + |\nabla V_H|^2) dx ds \\ & \leq C \left[\int_{\mathbf{R}^n} (|V_H|^2 + |(V_H)_t|^2 + |\nabla V_H|^2)(0) dx + (E_\rho^2 + \mathcal{M}^4) \int_0^t (1+s)^{-\frac{n}{2}-3} ds \right], \end{aligned}$$

for some positive μ . Denote

$$\mathcal{F}(t) = \int_{\mathbf{R}^n} (|V_H|^2 + |(V_H)_t|^2 + |\nabla V_H|^2) dx.$$

Then the above inequality gives

$$\mathcal{F}(t) + \mu \int_0^t \mathcal{F}(s) ds \leq C(\mathcal{F}(0) + (E_\rho^2 + \mathcal{M}^4) \int_0^t (1+s)^{-\frac{n}{2}-3} ds).$$

By using the Gronwall inequality, we have

$$\mathcal{F}(t) \leq C e^{-\mu t} (\mathcal{F}(0) + (E_\rho^2 + \mathcal{M}^4) \int_0^t e^{\mu s} (1+s)^{-\frac{n}{2}-3} ds).$$

Hence, we have

$$\begin{aligned} & \|V_H(t)\|_{H^1}^2 + \|(V_H)_t(t)\|_{L^2}^2 \\ & \leq e^{-\mu t} (\|V_H(0)\|_{H^1}^2 + \|(V_H)_t(0)\|_{L^2}^2) + C(E_\rho^2 + \mathcal{M}^4)(1+t)^{-\frac{n}{2}-3} \\ & \leq C(E_0^2 + \mathcal{M}^4)(1+t)^{-\frac{n}{2}-3}, \end{aligned}$$

where E_0 is defined in Theorem 4.2.

Next, we will derive the energy estimates on the higher order derivatives of the high frequency component, that is, $\int_{\mathbf{R}^n} |\partial^\alpha V_H|^2 + |\partial^\alpha (V_H)_t|^2 + |\nabla \partial^\alpha V_H|^2 dx$ for $0 < |\alpha| \leq k-1$. In the rest of this section, we assume $0 < |\alpha| \leq k-1$. The estimation can be obtained by induction on $|\alpha|$. Assume that

$$\int_{\mathbf{R}^n} (|\partial^\gamma V_H|^2 + |\partial^\gamma (V_H)_t|^2 + |\nabla \partial^\gamma V_H|^2) dx \leq C(E_0^2 + \mathcal{M}^4)(1+t)^{-\frac{n}{2}-(|\gamma|+3)}$$

holds for any multi-index γ with $|\gamma| < |\alpha|$, we want to prove

$$\int_{\mathbf{R}^n} (|\partial^\alpha V_H|^2 + |\partial^\alpha (V_H)_t|^2 + |\nabla \partial^\alpha V_H|^2) dx \leq C(E_0^2 + \mathcal{M}^4)(1+t)^{-\frac{n}{2}-(|\alpha|+3)}. \quad (5.3)$$

Taking $\partial^\alpha \tilde{\chi}$ on (2.5), neglecting the \pm sign without confusing, and integrating its product with $\partial^\alpha V_H$ and $\partial^\alpha (V_H)_t$ over \mathbf{R}^n respectively, we have

$$\frac{d}{dt} \int_{\mathbf{R}^n} \partial^\alpha V_H \partial^\alpha (V_H)_t dx - \int_{\mathbf{R}^n} |\partial^\alpha (V_H)_t|^2 dx - \int_{\mathbf{R}^n} \partial^\alpha V_H \Delta \partial^\alpha \tilde{\chi} (aV) dx + \frac{d}{dt} \int_{\mathbf{R}^n} \frac{1}{2} |\partial^\alpha V_H|^2 dx \quad (5.4)$$

$$= \int_{\mathbf{R}^n} \partial^\alpha V_H \tilde{\chi} \partial^\alpha F dx \mp \int_{\mathbf{R}^n} \partial^\alpha V_H \tilde{\chi} \partial^\alpha \operatorname{div}[(\bar{\rho}^\pm + V^\pm) \nabla \varphi],$$

and

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbf{R}^n} \frac{1}{2} |\partial^\alpha (V_H)_t|^2 dx - \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \Delta \tilde{\chi} \partial^\alpha (aV) dx + \int_{\mathbf{R}^n} |\partial^\alpha (V_H)_t|^2 dx \\ &= \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \partial^\alpha F dx \mp \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \partial^\alpha \operatorname{div}[(\bar{\rho}^\pm + V^\pm) \nabla \varphi]. \end{aligned} \quad (5.5)$$

For the third term on the left hand side of (5.4), we have

$$-\int_{\mathbf{R}^n} \partial^\alpha V_H \Delta \tilde{\chi} \partial^\alpha (aV) dx = \int_{\mathbf{R}^n} a (\partial^\alpha \nabla V_H)^2 dx - \int_{\mathbf{R}^n} \partial^\alpha V_H \nabla [\nabla \tilde{\chi} \partial^\alpha, a] V dx.$$

Since

$$\|\partial_x^\beta a\|_{L^\infty} \leq C E_\rho (1+t)^{-|\beta|/2},$$

it holds that

$$\left| \int_{\mathbf{R}^n} \partial^\alpha V_H \nabla [\nabla \tilde{\chi} \partial^\alpha, a] V dx \right| \leq \eta \int_{\mathbf{R}^n} |\partial^\alpha V_H|^2 dx + C_\eta E_\rho^2 \mathcal{M}^2 (1+t)^{-\frac{n}{2}-3-|\alpha|}.$$

Similarly, for the second term on the left hand side of (5.5), we have

$$\begin{aligned} -\int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \Delta \tilde{\chi} \partial^\alpha (aV) dx &= \frac{d}{dt} \int_{\mathbf{R}^n} \frac{a}{2} |\nabla \partial^\alpha V_H|^2 dx - \int_{\mathbf{R}^n} \frac{a_t}{2} |\nabla \partial^\alpha V_H|^2 dx \\ &\quad - \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \nabla [\nabla \partial^\alpha \tilde{\chi}, a] V dx, \end{aligned}$$

where

$$\left| \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \nabla [\nabla \tilde{\chi} \partial^\alpha, a] V dx \right| \leq \eta \int_{\mathbf{R}^n} |\partial^\alpha (V_H)_t|^2 dx + C_\eta E_\rho^2 \mathcal{M}^2 (1+t)^{-\frac{n}{2}-3-|\alpha|}.$$

For the terms $\int_{\mathbf{R}^n} \partial^\alpha V_H \tilde{\chi} \partial^\alpha F dx$ and $\int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \partial^\alpha F dx$ on the right hand side of (5.4) and (5.5), we only estimate the second one because the estimation on the first is easier. Notice that the estimation on the terms with derivatives of order less or equal to $|\alpha| + 1$ follows directly from the definition of \mathcal{M} in (3.4). Thus, we consider the terms with derivatives of order higher than $|\alpha| + 1$. Firstly, by using the expression (2.5) for F , we have

$$\begin{aligned} F &= \tilde{Q} + \Delta(\mathcal{P}_1(\bar{\rho}, V)V^2) \\ &= [(R_\rho)_t + R_\rho] - (1 + \partial_t)(V \bar{u}_1)_{x_1} - \operatorname{div}((\bar{\rho} + V)_t U) - \operatorname{div}((\bar{\rho} + V)H) + \Delta(\mathcal{P}_1(\bar{\rho}, V)V^2). \end{aligned} \quad (5.6)$$

Since (2.1) implies

$$\operatorname{div} U = -(\bar{\rho} + V)^{-1} (V_t + (\bar{u} + U) \cdot \nabla V + (U \cdot \nabla) \bar{\rho} + V \operatorname{div} \bar{u} - R_\rho), \quad (5.7)$$

substituting (5.7) in (5.6), by the definition of H , we have

$$F = (\bar{u} + U) \cdot \nabla ((\bar{u} + U) \cdot \nabla V) + \Delta(\mathcal{P}_1(\bar{\rho}, V)V^2) + \mathcal{R},$$

where \mathcal{R} denotes the remainder which contains derivatives of U and K with order at most 1. Thus, $\partial^\alpha \mathcal{R}$ has derivatives with order at most $|\alpha| + 1$ ($\leq k$). Then

$$\int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \partial^\alpha F dx = N_1 + N_2 + N_3,$$

with

$$\begin{aligned} N_1 &= \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \partial^\alpha ((\bar{u} + U) \cdot \nabla ((\bar{u} + U) \cdot \nabla V)) dx, \\ N_2 &= \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \partial^\alpha \Delta (\mathcal{P}_1(\bar{\rho}, V) V^2) dx, \\ N_3 &= \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \partial^\alpha \mathcal{R} dx. \end{aligned}$$

For N_1 , we have

$$\begin{aligned} N_1 &= \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} (\bar{u} + U) \cdot \nabla ((\bar{u} + U) \cdot \nabla \partial^\alpha V) dx + \{\dots\} \\ &= \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t (\bar{u} + U) \cdot \nabla ((\bar{u} + U) \cdot \nabla \partial^\alpha V_H) dx + \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t [(\bar{u} + U) \cdot \nabla)^2, \tilde{\chi}] \partial^\alpha V dx + \{\dots\} \\ &= -\frac{d}{dt} \int_{\mathbf{R}^n} \frac{1}{2} |(\bar{u} + U) \cdot \nabla \partial^\alpha (V_H)|^2 dx - \int_{\mathbf{R}^n} (\bar{u} + U)_t \cdot \nabla \partial^\alpha (V_H) (\bar{u} + U) \cdot \nabla \partial^\alpha (V_H) dx \\ &\quad - \int_{\mathbf{R}^n} \nabla (\bar{u} + U) \cdot \nabla \partial^\alpha (V_H)_t (\bar{u} + U) \cdot \nabla \partial^\alpha (V_H) dx + \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t [(\bar{u} + U) \cdot \nabla)^2, \tilde{\chi}] \partial^\alpha V dx + \{\dots\} \\ &=: -\frac{d}{dt} \int_{\mathbf{R}^n} \frac{1}{2} |(\bar{u} + U) \cdot \nabla \partial^\alpha (V_H)|^2 dx + N_{1,1} + N_{1,2} + N_{1,3} + \{\dots\}. \end{aligned}$$

Here and in the subsequent of this section, we use $\{\dots\}$ to denote the terms with derivatives of order at most $|\alpha| + 1$. It is easy to see that

$$|N_{1,1} + N_{1,2} + N_{1,3} + \{\dots\}| \leq C(E_0 + \mathcal{M}^3)(1+t)^{-\frac{n}{2}-(|\alpha|+3)}.$$

Then for N_2 , we have

$$\begin{aligned} N_2 &= \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \partial^\alpha \Delta (\mathcal{P}(\bar{\rho}, V) V^2) dx \\ &= \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \mathcal{P}'_V \partial^\alpha \Delta V V^2 + 2\partial^\alpha (V_H)_t \tilde{\chi} \mathcal{P}(\bar{\rho}, V) V \partial^\alpha \Delta V dx + \{\dots\} \\ &=: N_{2,1} + N_{2,2} + \{\dots\}. \end{aligned}$$

By noticing that

$$\begin{aligned} N_{2,1} &= \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \mathcal{P}'_V \partial^\alpha \Delta V V^2 dx \\ &= \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t (V^2 \mathcal{P}'_V) \partial^\alpha \Delta V_H dx + \int_{\mathbf{R}^n} \partial^\alpha (V_H)_t [\tilde{\chi}, \mathcal{P}'_V V^2] \partial^\alpha \Delta V_L dx, \\ &= -\frac{d}{dt} \int_{\mathbf{R}^n} (\mathcal{P}'_V V^2) |\nabla \partial^\alpha V_H|^2 dx + O_1, \end{aligned}$$

with

$$O_1 \leq \eta \int_{\mathbf{R}^n} |\partial^\alpha(V_H)|^2 dx + C_\eta(E_0 + \mathcal{M}^3)(1+t)^{-\frac{n}{2}-|\alpha|+3}.$$

Similarly, we have

$$N_{2,2} = -2 \frac{d}{dt} \int_{\mathbf{R}^n} (\mathcal{P}V) |\nabla \partial^\alpha V_H|^2 dx + O_2,$$

with

$$O_2 \leq \eta \int_{\mathbf{R}^n} |\partial^\alpha(V_H)|^2 dx + C_\eta(E_0 + \mathcal{M}^3)(1+t)^{-\frac{n}{2}-|\alpha|+3}.$$

We still need to consider the terms from the expansion of $\partial^\alpha \operatorname{div}[(\bar{\rho}^\pm + V^\pm) \nabla \varphi]$, $0 < |\alpha| \leq k-1$:

- If all the derivatives $\partial^\alpha \operatorname{div}$ are taken on $\bar{\rho}^\pm + V^\pm$, it can be bounded by the a priori assumption and the fact that $\nabla \varphi \in L^\infty$.
- If all the derivatives $\partial^\alpha \operatorname{div}$ are taken on $\nabla \varphi$ then

$$\partial^\alpha \operatorname{div} \nabla \varphi = \partial^\alpha K,$$

and recall the good decay properties of $\partial^\alpha K$ for $|\alpha| \leq k-1$ in Section 3, then we have better decay on these terms.

- Other terms can be estimated similarly.

Again to close the energy estimate, we now use the fact that

$$\int_{\mathbf{R}^n} |\nabla_x \partial^\alpha V_H|^2 dx \geq \epsilon \int_{\mathbf{R}^n} |\partial^\alpha V_H|^2 dx.$$

By integrating (5.4) and (5.5) over $[0, t]$ and multiplying (5.4) by some suitably chosen constant $0 < \lambda < 1$, the combination of above estimates give (5.3). Therefore, we have the following estimates on the high frequency component.

Theorem 5.1 *Under the assumption of Theorem 1.1, we have, for $|\alpha| \leq k-1$,*

$$\|\partial^\alpha V_H^\pm\|_{H^1} + \|\partial^\alpha (V_H^\pm)_t\|_{L^2} \leq C(E_0 + \mathcal{M}^{3/2})(1+t)^{-\frac{n}{4}-\frac{|\alpha|+3}{2}},$$

where E_0 and \mathcal{M} are defined in Theorem 4.2 and (3.4), respectively.

6 Proof of Theorem 1.1.

In the previous two sections, we obtain the following estimates on the low frequency component by using the approximate Green function and the high frequency component by using the energy method respectively,

$$\|\partial^\alpha V_L^\pm(t)\|_{L^p} \leq C(E_0 + \mathcal{M}^2)(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}, \quad |\alpha| \leq k, \quad (6.1)$$

and

$$\|\partial^\alpha V_H^\pm\|_{H^1} + \|\partial^\alpha (V_H^\pm)_t\|_{L^2} \leq C(E_0 + \mathcal{M}^{3/2})(1+t)^{-\frac{n}{4}-\frac{|\alpha|+3}{2}}, \quad |\alpha| \leq k-1. \quad (6.2)$$

It remains to combine (6.1) and (6.2) to close the a priori assumption (3.4).

Firstly, by taking $p = 2$ in (6.1) and combining with (6.2), we have

$$\|\partial^\alpha V^\pm(t)\|_{L^2} \leq C(E_0 + \mathcal{M}^{3/2})(1+t)^{-\frac{n}{4}-\frac{|\alpha|+1}{2}}, \quad |\alpha| \leq k.$$

Next, by using the Sobolev embedding theorem, from (6.2), we have, for $|\alpha| \leq k-2$,

$$\begin{aligned} \|\partial^\alpha V_H^\pm(t)\|_{L^\infty} &\leq \|\partial^\alpha V_H^\pm(t)\|_{H^2} \leq \|\partial^\alpha V_H^\pm(t)\|_{L^2} + \|\nabla \partial^\alpha V_H^\pm(t)\|_{H^1} \\ &\leq C(E_0 + \mathcal{M}^{3/2})(1+t)^{-\frac{n}{4}-\frac{|\alpha|+3}{2}}. \end{aligned}$$

Moreover, for $n = 3$, it holds that $-\frac{n}{4} - \frac{|\alpha|+3}{2} \leq -\frac{n}{2} - \frac{|\alpha|+1}{2}$. Thus,

$$\|\partial^\alpha V_H^\pm(t)\|_{L^\infty} \leq C(E_0 + \mathcal{M}^{3/2})(1+t)^{-\frac{n}{2}-\frac{|\alpha|+1}{2}}, \quad |\alpha| \leq k-2. \quad (6.3)$$

Then, the interpolation of (6.2) and (6.3) leads to

$$\|\partial^\alpha V_H^\pm(t)\|_{L^p} \leq C(E_0 + \mathcal{M}^{3/2})(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}, \quad |\alpha| \leq k-2. \quad (6.4)$$

Combining (6.1) with (6.4) then gives

$$\|\partial^\alpha V^\pm(t)\|_{L^p} \leq C(E_0 + \mathcal{M}^{3/2})(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}, \quad |\alpha| \leq k-2.$$

Now, we turn to estimate U^\pm by using the equation (2.7). Note that

$$U^\pm(x, t) = e^{-t} U^\pm(x, 0) + \int_0^t e^{-(t-s)} ((\bar{\rho}^\pm)^{-1} \nabla(a^\pm V^\pm) + \bar{H}^\pm \pm \nabla \varphi)(x, s) ds,$$

and it is easy to check that, for $|\gamma| \leq k-2$,

$$\begin{cases} \|\partial^\gamma \bar{H}^\pm(s)\|_{L^\infty} \leq C(E_0 + \mathcal{M}^2)(1+s)^{-(n+1+\frac{|\gamma|+1}{2})}, \\ \|\partial^\gamma ((\bar{\rho}^\pm)^{-1} \nabla_x(a^\pm V^\pm))(s)\|_{L^\infty} \leq C(E_\rho + \mathcal{M}_V)(1+s)^{-\frac{n}{2}-\frac{|\gamma|+2}{2}}, \\ \|\partial^\gamma \nabla \varphi\|_{L^\infty} \leq \|\partial^\gamma K\|_{H^1} \leq C(E_\rho + \mathcal{M}^2)(1+t)^{-\frac{5}{4}n-2-\frac{|\gamma|}{2}}, \end{cases}$$

thus,

$$\|\partial^\gamma U^\pm(t)\|_{L^\infty} \leq C(E_\rho + \mathcal{M}_V + \mathcal{M}^2)(1+t)^{-\frac{n}{2}-\frac{|\gamma|+2}{2}}, \quad |\gamma| \leq k-2. \quad (6.5)$$

Next, for the L^2 -norm, we use energy estimate. Multiply (2.7) by $\bar{\rho}^\pm U^\pm$ and integrate, we have

$$\int_{\mathbf{R}^n} \bar{\rho}^\pm U^\pm U_t^\pm dx + \int_{\mathbf{R}^n} \nabla(a^\pm V^\pm) \cdot U^\pm dx + \int_{\mathbf{R}^n} \bar{\rho}^\pm (U^\pm)^2 dx = \int_{\mathbf{R}^n} \bar{H}^\pm \bar{\rho}^\pm U^\pm dx \pm \int_{\mathbf{R}^n} \bar{\rho}^\pm \nabla \phi \cdot U^\pm dx. \quad (6.6)$$

Note

$$\int_{\mathbf{R}^n} \bar{\rho}^\pm U^\pm U_t^\pm dx = \frac{d}{dt} \int_{\mathbf{R}^n} \bar{\rho}^\pm (U^\pm)^2 dx - \int_{\mathbf{R}^n} \partial_t \bar{\rho}^\pm (U^\pm)^2 dx,$$

in which the second term is small, and

$$|\int_{\mathbf{R}^n} \nabla(a^\pm V^\pm) \cdot U^\pm dx| \leq \varepsilon \int_{\mathbf{R}^n} (U^\pm)^2 dx + C(\varepsilon) \mathcal{M}^2 (1+t)^{-\frac{n}{2}-(1+1)}, \quad (6.7)$$

in which the second term has the decay rate of $\|\nabla V^\pm\|_{L^2}^2$, (which is the key observation that U^\pm has better decay than V^\pm).

The first term on the right hand side of (6.6), the nonlinear term, can be estimated as in K , which has better decay than (6.7). For the last term with $\nabla\varphi$, we have

$$\begin{aligned} \left| \int_{\mathbf{R}^n} \bar{\rho}^\pm \nabla\phi \cdot U^\pm dx \right| &\leq \|\bar{\rho}^\pm\|_{L^3} \|\nabla\phi\|_{L^6} \|U^\pm\|_{L^2} \leq \|\bar{\rho}^\pm\|_{L^2}^{1/2} \|\bar{\rho}^\pm\|_{L^6}^{1/2} \|E\|_{L^2} \|U^\pm\|_{L^2} \\ &\leq \varepsilon \|U^\pm\|_{L^2}^2 + C(\varepsilon) \mathcal{M}^2 (1+t)^{-\frac{5}{4}n-2-\frac{1}{4}}. \end{aligned}$$

Perform same estimates for $\partial^\alpha(|\alpha| \leq k)$, and use the similar argument as for V^\pm , we can get

$$\|\partial^\alpha U^\pm\|_{L^2} \leq \mathcal{M}(1+t)^{-\frac{n}{4}-\frac{|\alpha|+2}{2}}, \quad |\alpha| \leq k. \quad (6.8)$$

Interpolate (6.5) and (6.8) to have

$$\|\partial^\alpha U^\pm\|_{L^p} \leq \mathcal{M}(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+2}{2}}, \quad |\alpha| \leq k-2.$$

Then combine the above estimates to get

Theorem 6.1 *Under the assumption of Theorem 1.1, if the initial data (V_0^\pm, U_0^\pm) satisfies that*

$$|\rho_+ - \rho_-| + \|\nu^\pm(\cdot, 0)\|_{L^2 \cap L^1} + \|\nu_t^\pm(\cdot, 0)\|_{L^2} + \|V_0^\pm\|_{H^k \cup L^1} + \|V_t^\pm(0)\|_{H^{k-1}} + \|U_0^\pm\|_{H^k} \leq \epsilon_0,$$

where $\epsilon_0 > 0$ is a small constant, then there exists a unique global classical solution $(V^\pm, U^\pm) \in C([0, \infty), H^k) \cap C^1((0, \infty), H^{k-1})$ to (2.3). Moreover, we have

$$\begin{cases} \|\partial_x^\gamma V^\pm\|_{L^p} \leq C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}, & |\gamma| \leq k-2, \\ \|\partial_x^\gamma V^\pm\|_{L^2} \leq C(1+t)^{-\frac{n}{4}-\frac{|\gamma|+1}{2}}, & |\gamma| = k-1, k, \\ \|\partial_x^\gamma U^\pm\|_{L^p} \leq C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+2}{2}}, & |r| \leq k-2, \\ \|\partial_x^\gamma U^\pm\|_{L^2} \leq C(1+t)^{-\frac{n}{4}-\frac{k+1}{2}}, & |\gamma| = k-1, k. \end{cases}$$

This theorem implies (3.4) then closed the a priori assumption, and then it yields the main results (i) and (ii) in Theorem 1.1.

Before concluding this paper, we point out that even though the above discussion is for the space dimension $n = 3$, other higher dimensional cases can be considered similarly.

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